



On Size-Biased Weighted Transmuted Weibull Distribution

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Abstract- This paper offers a new weighted distribution called size biased weighted transmuted weibull distribution, denoted by (SBWTWD). Various useful statistical properties of this distribution are derived in this paper such as, the cumulative distribution function, Reliability function, hazard rate, reversed hazard rate and the r th moment. Plots for the probability density function at different values of shape parameters are provided. The maximum likelihood estimators of the unknown parameters of the proposed distribution are obtained. One data set has been analyzed for illustrative purposes.

Keywords- Weighted distributions, Transmuted distribution, Weibull distribution, Maximum likelihood method.

I. INTRODUCTION

Adding an extra parameter to an existing family of distribution functions is very common in the statistical distribution theory. Often introducing an extra parameter brings more flexibility to a class of distribution functions and it can be very useful for data analysis purposes. Especially the weibull distribution and its generalizations in the literature attract the most of the researchers due to its wide range applications. The Weibull distribution includes the exponential and the Rayleigh distributions as sub models, the usefulness and applications of parametric distributions including Weibull, Rayleigh are seen in various areas including reliability, renewal theory, and branching processes which can be seen in papers by many authors such as in {[16], [17], [25]}. Different generalizations of the Weibull distribution are common in the literature as in {[4], [5], [21], [22], [28], [38]} and another generalization of the weibull distribution using the concept of weighted distributions is available as in {[6], [8], [19], [24], [30], [34], [36], [37]}.

The use and application of weighted distributions in research related to reliability, bio-medicine, ecology and several other areas are of tremendous practical importance in mathematics, probability and statistics. These distributions arise naturally as a result of observations generated from a stochastic process and recorded with some weight function. The concept of these distributions has been employed in wide variety applications in many fields of real life such as medicine, reliability, and survival analysis, analysis of family data, ecology and forestry. It can be traced to the work of Fisher [14] in connection with his studies on how method of ascertainment can influence the form of distribution of recorded observations.

Azzalini [1] was first to introduce a shape parameter to a normal distribution depending on a weight function which is called the skew-normal distribution. Different works on introducing shape parameters for other symmetric distributions are available in the literature, several properties and their inference procedures are discussed by several authors see for example in {[2], [3]}. On the other side, Recently several authors introduced shape parameters for non-symmetric distributions as be shown in {[7], [9], [10], [12], [13], [15],[18], [23], [26], [29], [32], [33], [35]}.

In this paper we construct the size biased weighted transmuted weibull distribution and the sub-models which are the special cases of our proposed distribution. Various useful statistical properties of this model are derived in the next sections. We also present a numerical example of the proposed distribution considering the real life data-set for illustrative purposes.

This paper is organized as follows. Section 2 defines some basic materials and in Section 3, we provide the derivation of PDF of the proposed model and some particular cases are obtained in Section 4. Section 5 discusses the different statistical properties of this model. Estimation of the unknown parameters of the proposed model by maximum likelihood method is carried out in Section 6. The real data-set has been analyzed in Sections 7 and section 8 gives some brief conclusion.

II. MATERIALS AND METHODS

Weighted distributions concept can be traced from the study of Fisher and Rao. Let X be a non-negative random variable with its probability density function (pdf), $f(x)$, then the pdf of the weighted random variable X^w is given by:

$$f^w(x) = \frac{w(x).f(x)}{E[w(X)]}, \quad 0 < E[w(X)] < \infty, \quad x > 0 \quad (1)$$

where, $f(x)$ is the pdf of the base distribution and the weight function $w(x)$ is a non-negative function, that may depend on the parameter. When the weight function depends on the length of units of interest, $w(x) = x$, the resulting distribution is called length-biased which finds various applications in biomedical areas such as early detection of a disease. Rao [27] also used this distribution in the study of human families and wild-life populations. In this case the pdf of a length-biased random variable is defined as:

$$f^{LB}(x) = \frac{x \cdot f(x)}{E[X]}, \quad x > 0, 0 < E[X] < \infty.$$

More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, when $w(x) = x^c$, $c > 0$, then the resulting distribution is called size-biased and the pdf of a size-biased random variable is defined as:

$$f^{SB}(x) = \frac{x^c \cdot f(x)}{E[X^c]}, \quad 0 < E[X^c] < \infty, c > 0.$$

This type of sampling is a generalization of length- biased sampling. In this paper we use this weight function, $w(x) = x^c$, considering the transmuted weibull distribution as baseline distribution to get a new weighted distribution.

According to the Quadratic Rank Transmutation Map (QRTM) approach proposed by Shaw and Buckley [31] a random variable X is said to have transmuted probability distribution if its cdf, $F_T(x)$ and pdf, $f_T(x)$ are given by:

$$F_T(x) = (1 + \alpha)F(x) - \alpha F(x)^2, \quad |\alpha| \leq 1,$$

and,

$$f_T(x) = f(x)[(1 + \alpha) - 2\alpha F(x)],$$

where, $F(x)$, $f(x)$, are the cdf, pdf of the base distribution, respectively and α is the transmuted, shape parameter. Then, the cdf and the pdf of the transmuted weibull distribution (TWD) are given as follow:

$$F_T(x) = \left[1 - e^{-\lambda x^\beta}\right] \left(1 + \alpha e^{-\lambda x^\beta}\right),$$

and,

$$f_T(x) = \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left[1 - \alpha + 2\alpha e^{-\lambda x^\beta}\right], \quad (2)$$

where, $\lambda > 0, \beta > 0$ are the scale, shape parameters respectively, the pdf, $f(x)$, and the cdf, $F(x)$, of the weibull distribution take the forms as follow:

$$f(x) = \lambda \beta x^{\beta-1} e^{-\lambda x^\beta}, \quad \lambda > 0, \beta > 0, x > 0, \text{ and}$$

$$F(x) = \left[1 - e^{-\lambda x^\beta}\right].$$

The distribution in equation (2) includes especially the transmuted exponential and transmuted Rayleigh distributions as special cases where $\beta = 1$ and $\beta = 2$, respectively.

III. DERIVATION OF THE SIZE BIASED WEIGHTED TRANSMUTED WEIBULL DISTRIBUTION

In this section, we derive the probability density function of size biased weighted transmuted weibull distribution. The plot of pdf of this distribution at various choices of shape parameters values can also be shown in this section. We can get the pdf of size biased weighted transmuted weibull distribution as follows:

When,

$$w(x) = x^c. \quad (3)$$

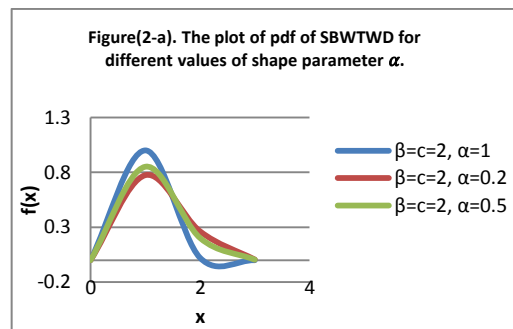
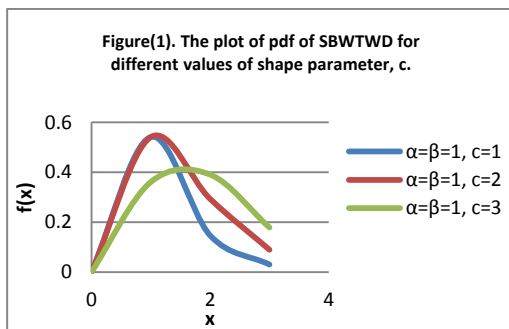
Substituting (3) and (2) in(1) then we get:

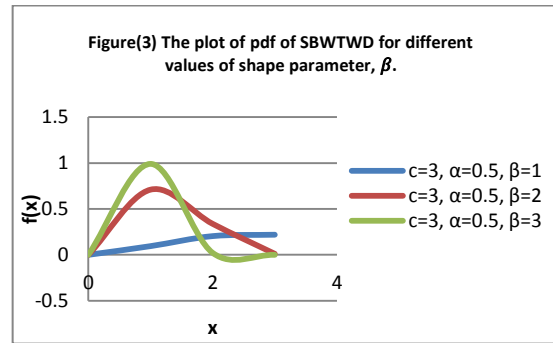
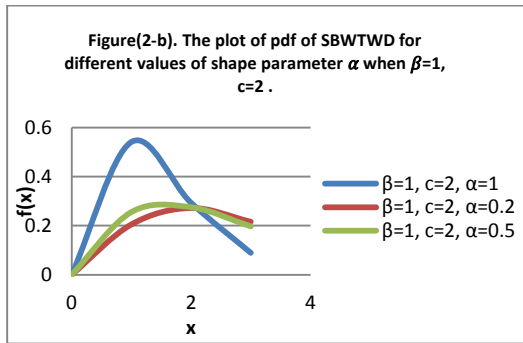
$$E(X^c) = \frac{\Gamma\left(\frac{c}{\beta} + 1\right) \left[1 - \alpha + \frac{\alpha}{(2)^\beta}\right]}{\lambda^{\frac{c}{\beta}}}.$$

Hence,

$$f^{SBWTWD}(x, \lambda, \alpha, \beta, c) = \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^\beta} \left[1 - \alpha + 2\alpha e^{-\lambda x^\beta}\right]}{\Gamma\left(\frac{c}{\beta} + 1\right) \left[1 - \alpha + \frac{\alpha}{(2)^\beta}\right]}, \quad x > 0, \lambda > 0, \beta > 0, c > 0, |\alpha| \leq 1 \quad (4)$$

The density function (4) can be known as size biased weighted transmuted weibull distribution, denoted by SBWTWD. Figures (1), [(2-a), (2-b)] and (3) represent the possible shapes of probability density function of the SBWTWD at different values of shape parameters c, α and β , respectively when the scale parameter, $\lambda = 1$.





IV. SOME PARTICULAR CASES OF SBWTWD

This section presents some sub-models that deduced from Equation (4) are:

Case1. Putting $c = 1$, the resulting distribution is length biased weighted transmuted weibull distribution (LBWTWD) given as:

$$f(x; \lambda, \alpha, \beta) = \frac{\lambda^{\frac{1}{\beta}+1} \beta^2 x^{\beta} e^{-\lambda x^{\beta}} [1 - \alpha + 2\alpha e^{-\lambda x^{\beta}}]}{\Gamma\left(\frac{1}{\beta}\right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{1}{\beta}}}\right]}, \quad x > 0, \lambda > 0, \beta > 0, |\alpha| \leq 1.$$

Case2. Putting $c = 1, \beta = 1$, the resulting distribution is length biased weighted transmuted exponential distribution (LBWTE) given as:

$$f(x; \lambda, \alpha) = \frac{2}{(2 - \alpha)} \lambda^2 x e^{-\lambda x} [1 - \alpha + 2\alpha e^{-\lambda x}], \quad x > 0, \lambda > 0, |\alpha| \leq 1$$

Case3. Putting $\alpha = 0$, the resulting distribution is sized biased weighted weibull distribution (SBWWD) given as:

$$f(x; \lambda, \beta, c) = \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^{\beta}}}{\Gamma\left(\frac{c}{\beta} + 1\right)}, \quad x > 0, \lambda > 0, \beta > 0, c > 0.$$

Case4. Putting $\alpha = 1$, the resulting distribution is sized biased weighted weibull distribution (SBWWD) given as:

$$f(x; \lambda, \beta, c) = \frac{(2\lambda)^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-2\lambda x^{\beta}}}{\Gamma\left(\frac{c}{\beta} + 1\right)}, \quad x > 0, \lambda > 0, \beta > 0, c > 0.$$

Case5. Putting $\alpha = 1, \beta = 1, c = 1$, the resulting distribution is length biased weighted exponential distribution (LBWED) given as:

$$f(x; \lambda) = (2\lambda)^2 x e^{-2\lambda x}, \quad x > 0, \lambda > 0.$$

Case6. Putting $\alpha = 1, \beta = 2, c = 1$, the resulting distribution is length biased weighted Rayleigh distribution (LBWRD) given as:

$$f(x; \lambda) = \frac{2^{\frac{5}{2}} (\lambda)^{\frac{3}{2}} x^2 e^{-2\lambda x^2}}{\Gamma\left(\frac{3}{2}\right)}, \quad x > 0, \lambda > 0.$$

Case7. Putting $\alpha = 1, c = 1$, the resulting distribution is length biased weighted weibull distribution (LBWWD) given as:

$$f(x; \lambda, \beta) = \frac{(2\lambda)^{\frac{1}{\beta}+1} \beta x^{\beta} e^{-2\lambda x^{\beta}}}{\Gamma\left(\frac{1}{\beta} + 1\right)}, \quad x > 0, \lambda > 0, \beta > 0.$$

Case8. Putting $\alpha = 0, c = 1, \beta = 1$, the resulting distribution is length biased weighted exponential distribution (LBWED) given as:

$$f(x; \lambda) = \lambda^2 x e^{-\lambda x}, \quad x > 0, \lambda > 0.$$

Case9. Putting $c = 0$, the resulting distribution is transmuted weibull distribution (TWD) given as:

$$f(x; \lambda, \alpha, \beta) = \lambda \beta x^{\beta-1} e^{-\lambda x^{\beta}} [1 - \alpha + 2\alpha e^{-\lambda x^{\beta}}], \quad x > 0, \lambda > 0, \beta > 0, |\alpha| \leq 1$$

Case10. Putting $c = 0, \alpha = 0$, the resulting distribution is weibull distribution (WD) given as:

$$f(x; \lambda, \beta) = \lambda \beta x^{\beta-1} e^{-\lambda x^{\beta}}, \quad x > 0, \lambda > 0, \beta > 0.$$

Case11. Putting $c = 0, \beta = 1$, the resulting distribution is transmuted exponential distribution (TED) given as:

$$f(x; \lambda, \alpha) = \lambda e^{-\lambda x} [1 - \alpha + 2\alpha e^{-\lambda x}], \quad x > 0, \lambda > 0, |\alpha| \leq 1.$$

Case12. Putting $c = 1, \beta = 2$, the resulting distribution is length biased weighted transmuted Rayleigh distribution (LBWTRD) given as:

$$f(x; \lambda, \alpha) = \frac{2\lambda^{\frac{3}{2}}x^2 e^{-\lambda x^2} [1 - \alpha + 2\alpha e^{-\lambda x^2}]}{\Gamma\left(\frac{3}{2}\right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{1}{2}}}\right]}, \quad x > 0, \lambda > 0, |\alpha| \leq 1.$$

Case13. Putting $c = 0, \beta = 2$, the resulting distribution is transmuted Rayleigh distribution (TRD) given as:

$$f(x; \lambda, \alpha) = 2\lambda x e^{-\lambda x^2} [1 - \alpha + 2\alpha e^{-\lambda x^2}], \quad x > 0, \lambda > 0, |\alpha| \leq 1.$$

Case14. Putting $c = 0, \beta = 2, \alpha = 0$, the resulting distribution is Rayleigh distribution (RD) given as:

$$f(x; \lambda) = 2\lambda x e^{-\lambda x^2}, \quad x > 0, \lambda > 0.$$

Case15. Putting $\alpha = 0, c = 0, \beta = -2$ and multiplying by (-1) , this model gives the inverse Rayleigh distribution (IRD) given as:

$$f(x; \lambda) = 2\lambda x^{-3} e^{-\lambda x^{-2}}, \quad x > 0, \lambda > 0.$$

Case16. Putting $\alpha = 1, c = 0, \beta = -2$ and multiplying by (-1) , this model gives the inverse Rayleigh distribution (IRD) given as:

$$f(x; \lambda) = 2(2\lambda)x^{-3} e^{-2\lambda x^{-2}}, \quad x > 0, \lambda > 0.$$

Case17. Putting $\alpha = 0, c = 1$, the resulting distribution is length biased weighted weibull distribution (LBWWD) given as:

$$f(x; \lambda, \beta) = \frac{\lambda^{\frac{1}{\beta}+1} \beta x^\beta e^{-\lambda x^\beta}}{\Gamma\left(\frac{1}{\beta} + 1\right)}, \quad x > 0, \lambda > 0, \beta > 0.$$

Case18. Putting $c = 1, \beta = 2, \alpha = 0$, the resulting distribution is length biased weighted Rayleigh distribution (LBWRD) given as:

$$f(x; \lambda) = \frac{2\lambda^{\frac{3}{2}}x^2 e^{-\lambda x^2}}{\Gamma\left(\frac{3}{2}\right)}, \quad x > 0, \lambda > 0.$$

Case19. Putting $c = 0, \alpha = 1$, the resulting distribution is weibull distribution (WD) given as:

$$f(x; \lambda, \beta) = 2\lambda \beta x^{\beta-1} e^{-2\lambda x^\beta}, \quad x > 0, \lambda > 0, \beta > 0.$$

Case20. Putting $c = 0, \beta = 1, \alpha = 1$, the resulting distribution is exponential distribution (ED) given as:

$$f(x; \lambda) = 2\lambda e^{-2\lambda x}, \quad x > 0, \lambda > 0.$$

Case21. Putting $c = 0, \beta = 2, \alpha = 1$, the resulting distribution is Rayleigh distribution (RD) given as:

$$f(x; \lambda) = 2(2\lambda)x e^{-2\lambda x^2}, \quad x > 0, \lambda > 0.$$

Case22. Putting $c = 0, \beta = 1, \alpha = 0$, the resulting distribution is exponential distribution (ED) given as:

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0.$$

Case23. Putting $\beta = 1$, the resulting distribution is size biased weighted transmuted exponential distribution (SBWTED) given as:

$$f(x; \lambda, \alpha, c) = \frac{\lambda^{c+1} x^c e^{-\lambda x} [1 - \alpha + 2\alpha e^{-\lambda x}]}{\Gamma(c+1) \left[1 - \alpha + \frac{\alpha}{(2)^c}\right]}, \quad x > 0, \lambda > 0, c > 0, |\alpha| \leq 1$$

Case24. Putting $\beta = 2$, the resulting distribution is size biased weighted transmuted Rayleigh distribution (SBWTRD) given as:

$$f(x; \lambda, \alpha, c) = \frac{2\lambda^{\frac{c}{2}+1} x^{c+1} e^{-\lambda x^2} [1 - \alpha + 2\alpha e^{-\lambda x^2}]}{\Gamma\left(\frac{c}{2} + 1\right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{2}}}\right]}, \quad x > 0, \lambda > 0, c > 0, |\alpha| \leq 1.$$

V. THE STATISTICAL PROPERTIES OF SBWTWD

In this section, we present some basic statistical properties of SBWTWD including, the cumulative distribution function (CDF), reliability function, hazard function and the reverse hazard function, r th moment, the mean, variance and order statistics as follow:

i. The CDF of SBWTWD is defined as:

$$F^{SBWTWD}(x) = \int_0^x f^{SBWTWD}(t) dt.$$

Therefore, The CDF of SBWTWD is given as:

$$F^{SBWTWD}(x, \lambda, \alpha, \beta, c) = \frac{\gamma\left(\frac{c}{\beta} + 1, \lambda x^\beta\right)}{\Gamma\left(\frac{c}{\beta} + 1\right)},$$

where, $\gamma(s, x)$ is the lower incomplete gamma function defined as:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

ii. The Reliability (Survival) function of SBWTWD is defined as:

$$R^{SBWTWD}(x, \lambda, \alpha, \beta, c) = 1 - F^{SBWTWD}(x, \lambda, \alpha, \beta, c),$$

$$= 1 - \frac{\gamma\left(\frac{c}{\beta} + 1, \lambda x^\beta\right)}{\Gamma\left(\frac{c}{\beta} + 1\right)}.$$

Table (1) contains the values of survival function of SBWTWD. Looking at this table we can see that the survival probability of the distribution increases with increase in the value of c for a holding x, λ and β at a fixed level. Further, from the table we can see that; for fixed c, λ and β ; the survival probability decreases with increase in x .

Table 1: Survival function of SBWTWD

$\lambda = 1, \beta = 1$					
x	c				
	1	1.2	1.4	1.6	1.8
0.1	0.995	0.998	0.999	0.999	1
0.2	0.982	0.99	0.994	0.996	0.998
0.3	0.963	0.976	0.985	0.991	0.994
0.4	0.938	0.958	0.972	0.981	0.988
0.5	0.91	0.936	0.955	0.969	0.979
0.6	0.878	0.91	0.935	0.953	0.967
0.7	0.844	0.882	0.912	0.935	0.953
0.8	0.809	0.852	0.887	0.914	0.936
0.9	0.772	0.82	0.859	0.891	0.917
1	0.736	0.787	0.83	0.866	0.896
1.1	0.699	0.753	0.8	0.84	0.873

iii. The hazard rate function of the random variable X^w follows SBWTWD is defined by:

$$H^{SBWTWD}(x, \lambda, \alpha, \beta, c) = \frac{f^{SBWTWD}(x)}{1 - F^{SBWTWD}(x)} = \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^\beta} [1 - \alpha + 2\alpha e^{-\lambda x^\beta}]}{\left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right] \left[\Gamma\left(\frac{c}{\beta} + 1\right) - \gamma\left(\frac{c}{\beta} + 1, \lambda x^\beta\right)\right]}, \quad x > 0.$$

iv. The reversed hazard rate function of the random variable X^w follows SBWTWD is given as:

$$\tilde{H}^{SBWTWD}(x, \lambda, \alpha, \beta, c) = \frac{f^{SBWTWD}(x)}{F^{SBWTWD}(x)} = \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^\beta} [1 - \alpha + 2\alpha e^{-\lambda x^\beta}]}{\gamma\left(\frac{c}{\beta} + 1, \lambda x^\beta\right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]}, \quad x > 0.$$

v. The r th moment of the random variable X^w follows SBWTWD is given as:

$$\dot{M}_r^{SBWTWD} = \frac{\Gamma\left(\frac{r+c}{\beta} + 1\right)}{\Gamma\left(\frac{c}{\beta} + 1\right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} \left(\frac{\lambda}{(2)^{\frac{c}{\beta}}}\right)^r \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{r+c}{\beta}}}\right], \quad r = 1, 2, 3, ..$$

Or \dot{M}_r^{SBWTWD} can be written as:

$$\dot{M}_r^{SBWTWD} = \frac{\Gamma_r k_r}{\Gamma k a^r},$$

where, $\Gamma_r = \Gamma\left(\frac{r+c}{\beta} + 1\right)$, $k_r = \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{r+c}{\beta}}}\right]$, $\Gamma = \Gamma\left(\frac{c}{\beta} + 1\right)$, $k = \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]$ and $a = (\lambda)^{\frac{1}{\beta}}$.

For the case, $r = 1, 2, 3, 4$ we have,

$$\dot{\mu}_1 = \frac{\Gamma_1 k_1}{\Gamma k a}, \quad \dot{\mu}_2 = \frac{\Gamma_2 k_2}{\Gamma k a^2}, \quad \dot{\mu}_3 = \frac{\Gamma_3 k_3}{\Gamma k a^3}, \quad \dot{\mu}_4 = \frac{\Gamma_4 k_4}{\Gamma k a^4}.$$

vi. The variance of the random variable X^w follows SBWTWD is given as::

$$\sigma^2^{SBWTWD} = \dot{\mu}_2 - \dot{\mu}_1^2 = \frac{\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2}{\Gamma^2 k^2 a^2}.$$

vii. The first central moments of SBWTWD are given by:

$$\mu_1 = 0, \quad \mu_2 = \sigma^2 = \dot{\mu}_2 - \dot{\mu}_1^2, \quad \mu_3 = \dot{\mu}_3 - 3\dot{\mu}_1 \dot{\mu}_2 + 2\dot{\mu}_1^3, \quad \mu_4 = \dot{\mu}_4 - 4\dot{\mu}_1 \dot{\mu}_3 + 6\dot{\mu}_1^2 \dot{\mu}_2 - 3\dot{\mu}_1^4,$$

where,

$$\mu_3 = \frac{\Gamma_3 k_3}{\Gamma \cdot k a^3} - 3 \frac{\Gamma_1 k_1 \Gamma_2 k_2}{(\Gamma \cdot k)^2 a^3} + 2 \frac{[\Gamma_1 k_1]^3}{[\Gamma \cdot k a]^3} = \frac{(\Gamma k)^2 \Gamma_3 k_3 - 3\Gamma k \Gamma_1 k_1 \Gamma_2 k_2 + 2[\Gamma_1 k_1]^3}{[\Gamma k a]^3},$$

$$\begin{aligned} \mu_4 &= \frac{\Gamma_4 k_4}{\Gamma k a^4} - 4 \frac{\Gamma_1 k_1 \Gamma_3 k_3}{(\Gamma k)^2 a^4} + 6[\Gamma_1 k_1]^2 \frac{\Gamma_2 k_2}{(\Gamma k)^3 a^4} - 3 \left[\frac{\Gamma_1 k_1}{\Gamma k a} \right]^4 \\ &= \frac{[\Gamma k]^3 \Gamma_4 k_4 - 4(\Gamma k)^2 \Gamma_1 k_1 \Gamma_3 k_3 + 6\Gamma k [\Gamma_1 k_1]^2 \Gamma_2 k_2 - 3[\Gamma_1 k_1]^4}{[\Gamma k a]^4} \end{aligned}$$

viii. The coefficient of variation is given as:

$$CV = \frac{\sigma}{\mu_1} = [\Gamma k a] \frac{\sqrt{\frac{\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2}{\Gamma^2 k^2 a^2}}}{\Gamma_1 k_1} = \frac{\sqrt{\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2}}{\Gamma_1 k_1}$$

ix. Coefficient of Skewness (SK) is given by:

$$SK = \frac{\mu_3}{\sigma^3} = \frac{(\Gamma k)^2 \Gamma_3 k_3 - 3\Gamma k \Gamma_1 k_1 \Gamma_2 k_2 + 2[\Gamma_1 k_1]^3}{\left[\frac{\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2}{\Gamma^2 k^2 a^2} \right]^{\frac{3}{2}} [\Gamma k a]^3} = \frac{(\Gamma k)^2 \Gamma_3 k_3 - 3\Gamma k \Gamma_1 k_1 \Gamma_2 k_2 + 2[\Gamma_1 k_1]^3}{[\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2]^{\frac{3}{2}}}$$

x. Coefficient of Kurtosis (ku) is given by:

$$Ku = \frac{\mu_4}{\sigma^4} - 3 = \frac{[\Gamma k]^3 \Gamma_4 k_4 - 4(\Gamma k)^2 \Gamma_1 k_1 \Gamma_3 k_3 + 6\Gamma k [\Gamma_1 k_1]^2 \Gamma_2 k_2 - 3[\Gamma_1 k_1]^4}{[\Gamma k \Gamma_2 k_2 - [\Gamma_1 k_1]^2]^2} - 3.$$

xi. The mode is the value of the random variable x which makes the pdf is a maximum.

Taking logarithm of the pdf of SBWTWD as:

$$\begin{aligned} \log_e f^{SBWTWD}(x, \lambda, \alpha, \beta, c) &= \left(\frac{c}{\beta} + 1 \right) \log_e \lambda + \log_e \beta + (c + \beta - 1) \log_e x - \lambda x^\beta + \log_e [1 - \alpha + 2\alpha e^{-\lambda x^\beta}] - \log_e \Gamma \left(\frac{c}{\beta} + 1 \right) \\ &\quad - \log_e \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}} \right]. \end{aligned}$$

$$\frac{\partial \log_e f^{SBWTWD}(x, \lambda, \alpha, \beta, c)}{\partial x} = \frac{(c + \beta - 1)}{x} - \lambda \beta x^{\beta-1} - \frac{2\alpha \lambda \beta x^{\beta-1} e^{-\lambda x^\beta}}{[1 - \alpha + 2\alpha e^{-\lambda x^\beta}]} \quad (5)$$

The mode of the SBWTWD is obtained by solving the equation (5) with respect to x as follow:

$$\frac{(c + \beta - 1)}{x} - \lambda \beta x^{\beta-1} - \frac{2\alpha \lambda \beta x^{\beta-1} e^{-\lambda x^\beta}}{[1 - \alpha + 2\alpha e^{-\lambda x^\beta}]} = 0. \quad (6)$$

By solving the nonlinear equation (6), can be calculated the mode of the SBWTWD.

xii. The order statistics have great importance in life testing and reliability analysis. Let X_1, X_2, \dots, X_n be random variables and its ordered values is denoted as x_1, x_2, \dots, x_n . The pdf of order statistics is obtained using the below function:

$$f_{s:n}(x) = \frac{n!}{(s-1)!(n-s)!} f(x)[F(x)]^{s-1}[1-F(x)]^{n-s}. \quad (7)$$

To obtain the smallest value in random sample of size n put $s = 1$ in (7), then the pdf of smallest order statistics is given by

$$f_{1:n}(x) = n f(x) [1 - F(x)]^{n-1}.$$

Therefore, the pdf of smallest order statistics for the SBWTWD is:

$$f_{1:n}(x) = n \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^\beta} [1 - \alpha + 2\alpha e^{-\lambda x^\beta}]}{\Gamma \left(\frac{c}{\beta} + 1 \right) \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}} \right]} \left[1 - \frac{\gamma \left(\frac{c}{\beta} + 1, \lambda x^\beta \right)}{\Gamma \left(\frac{c}{\beta} + 1 \right)} \right]^{n-1}, \quad \lambda, \beta > 0, x > 0.$$

To obtain the largest value in random sample of size n put $s = n$ in (7), then the pdf of order statistics is given by:

$$f_{n:n}(x) = n f(x) [F(x)]^{n-1}.$$

Therefore, the pdf of largest order statistics for the SBWTWD is:

$$f_{n:n}(x) = n \frac{\lambda^{\frac{c}{\beta}+1} \beta x^{c+\beta-1} e^{-\lambda x^\beta} [1 - \alpha + 2\alpha e^{-\lambda x^\beta}]}{\left[\Gamma \left(\frac{c}{\beta} + 1 \right) \right]^n \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}} \right]} \left[\gamma \left(\frac{c}{\beta} + 1, \lambda x^\beta \right) \right]^{n-1}, \quad x > 0.$$

VI. MAXIMUM LIKELIHOOD ESTIMATION OF THE SBWTWD

Let x_1, x_2, \dots, x_n be an independent random sample from the SBWTWD, then the likelihood function, $L(x; \lambda, \beta, \alpha, c)$, of SBWTWD is given by:

$$L(x; \lambda, \beta, \alpha, c) = \prod_{i=1}^n f^{SBWTWD}(x, \lambda, \alpha, \beta, c). \quad (8)$$

Substituting from (4) into (8), we have,

$$L(x; \lambda, \beta, \alpha, c) = \frac{\lambda^{n(\frac{c}{\beta}+1)} \beta^n}{\left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]^n \left[\Gamma\left(\frac{c}{\beta} + 1\right)\right]^n} \prod_{i=1}^n x_i^{c+\beta-1} e^{-\lambda \sum_{i=1}^n x_i^\beta} \prod_{i=1}^n \left[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}\right].$$

So, logarithm likelihood function $\log_e L(x; \lambda, \beta, \alpha, c)$, is given as:

$$\begin{aligned} \log_e L(x; \lambda, \beta, \alpha, c) &= \frac{nc}{\beta} \log_e \lambda + n \log_e \lambda \\ &+ n \log_e \beta - n \log_e \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right] - n \log_e \Gamma\left(\frac{c}{\beta} + 1\right) + (c + \beta - 1) \sum_{i=1}^n \log_e x_i - \lambda \sum_{i=1}^n x_i^\beta \\ &+ \sum_{i=1}^n \log_e \left[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}\right]. \end{aligned} \tag{9}$$

Differentiating (9) with respect to λ, β, α , and c , respectively, as follows:

$$\frac{\partial \log_e L(x; \lambda, \beta, \alpha, c)}{\partial \lambda} = \frac{nc}{\lambda \beta} + \frac{n}{\lambda} - \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n \frac{(2\alpha x_i^\beta) e^{-\lambda x_i^\beta}}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]}, \tag{10}$$

$$\begin{aligned} \frac{\partial \log_e L(x; \lambda, \beta, \alpha, c)}{\partial \beta} &= \frac{n}{\beta} - \frac{nc}{\beta^2} \log_e \lambda - \frac{n\alpha c \ln 2}{(2)^{\frac{c}{\beta}} \beta^2 \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} + \frac{nc}{\beta^2} \psi\left(\frac{c}{\beta} + 1\right) + \sum_{i=1}^n \log_e x_i \\ &- \lambda \sum_{i=1}^n (x_i^\beta \ln x_i) - \sum_{i=1}^n \frac{2\alpha \lambda e^{-\lambda x_i^\beta} (x_i^\beta \ln x_i)}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]}, \end{aligned} \tag{11}$$

where, $\psi\left(\frac{c}{\beta} + 1\right)$ is the digamma function.

$$\frac{\partial \log_e L(x; \lambda, \beta, \alpha, c)}{\partial \alpha} = \frac{n \left[1 - \frac{1}{(2)^{\frac{c}{\beta}}}\right]}{\left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} - \sum_{i=1}^n \frac{(1 - 2e^{-\lambda x_i^\beta})}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]}, \tag{12}$$

$$\frac{\partial \log_e L(x; \lambda, \beta, \alpha, c)}{\partial c} = \frac{n}{\beta} \log_e \lambda + \frac{n\alpha \ln 2}{\beta 2^{\frac{c}{\beta}} \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} - \frac{n}{\beta} \psi\left(\frac{c}{\beta} + 1\right) + \sum_{i=1}^n \log_e x_i. \tag{13}$$

Setting the equations(10), (11), (12)and (13)equal to zero, we have the following equations:

$$\frac{nc}{\lambda \beta} + \frac{n}{\lambda} - \sum_{i=1}^n x_i^\beta - \sum_{i=1}^n \frac{(2\alpha x_i^\beta) e^{-\lambda x_i^\beta}}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]} = 0, \tag{14}$$

$$\begin{aligned} \frac{n}{\beta} - \frac{nc}{\beta^2} \log_e \lambda - \frac{n\alpha c \ln 2}{(2)^{\frac{c}{\beta}} \beta^2 \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} + \frac{nc}{\beta^2} \psi\left(\frac{c}{\beta} + 1\right) + \sum_{i=1}^n \log_e x_i \\ - \lambda \sum_{i=1}^n (x_i^\beta \ln x_i) - \sum_{i=1}^n \frac{2\alpha \lambda e^{-\lambda x_i^\beta} (x_i^\beta \ln x_i)}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]} = 0, \end{aligned} \tag{15}$$

$$\frac{n \left[1 - \frac{1}{(2)^{\frac{c}{\beta}}}\right]}{\left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} - \sum_{i=1}^n \frac{(1 - 2e^{-\lambda x_i^\beta})}{[1 - \alpha + 2\alpha e^{-\lambda x_i^\beta}]} = 0, \tag{16}$$

$$\frac{n}{\beta} \log_e \lambda + \frac{n\alpha \ln 2}{\beta 2^{\frac{c}{\beta}} \left[1 - \alpha + \frac{\alpha}{(2)^{\frac{c}{\beta}}}\right]} - \frac{n}{\beta} \psi\left(\frac{c}{\beta} + 1\right) + \sum_{i=1}^n \log_e x_i = 0. \tag{17}$$

We can get MLEs of the unknown parameters by solving the equations(14),(15), (16)and (17)to estimate the parameters λ, β, α and c using numerical technique methods such as newton Raphson method because it is not possible to solve these equations analytically. By taking the second partial derivatives of (10), (11), (12)and (13) the Fisher's information matrix can be obtained by taking the negative expectations of the second partial derivatives. The inverse of the Fisher's information matrix is the variance covariance matrix of the maximum likelihood estimators.

VII. APPLICATION

In this section, we provide an application of the proposed distribution to show the importance of the new model. The data set (gauge lengths of 10 mm) from Kundu and Raqab [20]. This data set consists of, 63 observations as the following:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

This data set is previously studied by Afify et al [11] to fit the transmuted weibull lomax distribution. We fit both transmuted weibull (TW) and size biased weighted transmuted weibull (SBWTW) distributions to the subject data. We also estimate the parameters λ, α, β and c using Newton-Raphson method by taking the initial estimates $\hat{\lambda}_0 = 2.5321$, $\hat{\beta}_0 = 1.1577$, $\hat{\alpha}_0 = 0.9$, $\hat{c}_0 = 0.216$ and the estimated values of the parameters can be shown in table 2. To see which one of these models is more appropriate to fit the data set, we calculate Akaike Information Criterion (AIC), the Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The best distribution corresponds to lower for $(-2)\log$ -likelihood, AIC, BIC, and CAIC statistics values, where,

$$AIC = -2\ell + 2k,$$

$$CAIC = -2\ell + 2kn/(n - k - 1), \text{ and } BIC = -2\ell + k(\ln n)$$

where ℓ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. These numerical results are obtained using the MATH- CAD PROGRAM.

Table 2. The Estimated Values of the Parameters

Model	Parameters estimates values			
	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	\hat{c}
TW	0.0041	4.718	-0.2542	—
SBWTW	0.2564	2.5007	0.6819	9.6564

Table (2) contains the estimated values of the parameters for the (TWD) and SBWTWD.

Table 3. The Statistics (-2ℓ) , AIC, BIC and CAIC for Gauge Lengths of 10 MM Data Set.

Models	-2ℓ	AIC	BIC	CAIC
TW	124.414	130.414	136.843	130.82
SBWTW	116.861	124.861	133.433	125.551

Table (3) contains the values of (-2ℓ) , AIC, BIC and CAIC statistics. We note that the SBWTW model gives the lowest values for (-2ℓ) , AIC, BIC and CAIC statistics so that SBWTWD leads to a better fit to these data than TWD.

VIII. CONCLUSION

In this paper we propose a new four-parameter model, called size biased weighted transmuted weibull distribution which is a generalization of transmuted weibull distribution. We present some of its statistical properties. The new distribution is very flexible model that approaches to different life time distributions when its parameters are changed. We discuss maximum likelihood estimation. We consider Akaike Information Criterion (AIC), the Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC) statistics to compare the model with transmuted weibull model. An application of the size biased weighted transmuted weibull distribution to real data shows that the proposed distribution can be used quite effectively to provide better fits than the transmuted-weibull distribution.

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