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Numerical Treatment of the Nonlocal Solution of Hammerstein –Volterra Integral Equation with Continuous Kernels

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Abstract: ***In this paper, the existence of a unique solution of Hammerstein –Volterra integral equation (H-VIE) of the second kind is considered and proved, under certain conditions, using Banach fixed point theorem. Moreover, a suitable numerical method, Collocation method, is used to reduce the H-VIE to a nonlocal nonlinear algebraic system (nonlocal NAS). The existence of a unique solution of the nonlocal NAS is considered. Many numerical results are calculated, when the nonlocal term is neglected in the linear case and in the nonlinear case, and the error estimate, in each case, is computed. In addition, some special cases are derived and computed, in this work, when the memory takes different cases.***

Key Wards: Nonlocal Hammerstein –Volterra integral equation, Banach fixed point theorem, Collocation method, nonlocal nonlinear algebraic system.
MSC (2010):45B05; 45G10;65R .

I. INTRODUCTION

In recent years, the theory of integral equations has a considerable interest of many authors in numerous diverse fields, such as; ordinary and partial differential equations, viscodynamics fluid, contact problems in the theory of elasticity, mixed boundary problems in mathematical physics, biologic, chemistry and processes engineering, see Green [1], Hochstadt [2], Kanwal [3] and Schiavone et al. [4]. For interested reader, see Abdou [5, 6], Bai et al. [7] and Tari et al. [8].

The references Burton [9], Linz [10], Muskhelishvili [11], Aleksandrov and Covalence [12], Reihani et al. [13] and Cardone et al. [14] contain many different methods to obtain the solution of **IE** analytically and numerically. At the same time, numerical methods have played an important role in solving the **IE** of different kinds and types. It is our belief that an important component in the choice of a numerical method to solve a particular scientific problem is its mathematical reliability, i.e. knowledge of its convergence and stability properties. These properties can only be established by mathematical techniques. Among these numerical methods are: collocation method, Galerkin method, fast method, block by block method, Gerosonli's method, least square method, Product Nystrom method, and Toeplitz matrix method. The books edited by Golberg [15], Delves and Mohamed [16], Atkinson [17, 18], and Baker [19] contain these methods. More information for other methods, see [20-25].

Consider the nonlocal **H-VIE**:

$$\mu\,\phi(t) = f\left(t, H_1\big(t,\phi(t)\big)\right) + \lambda_1 \int_0^1 k(t,y) H_2\big(y,\phi(y)\big)dy + \lambda_2 \int_0^t F(t,\tau) H_3\big(\tau,\phi(\tau)\big)d\tau. \qquad (1.1)$$

of the second kind with continuous kernels in the space $L_1[0,1]$, $t < 1$. Here, $f\left(t, H_1\big(t,\phi(t)\big)\right)$ is called the free term and $H_1\big(t,\phi(t)\big)$ is known as the memory of the **H-VIE** (1.1), $H_s\big(x,\phi(x)\big)$; $s = 2,3$, are known functions. The two kernels (t,y) , $F(t,\tau)$ of **HI** term and of **VI** term, respectively, are known continuous functions, while $\phi(t)$ is unknown function in the space $L_1[0,1]$, $t < 1$, and it represents the solution of the **H-VIE** (1.1).The constant μ defines the kind of the integral equation , and the constants λ_i ; $i = 1,2$, have physical meanings.

In this paper, the existence of a unique solution of the nonlocal **H-VIE** (1.1) is considered and proved, under certain conditions, using Banach fixed point theorem. Moreover, a numerical method, Collocation method, is used to reduce the **H-VIE** (1.1) to a nonlocal **NAS.** The nonlocal **NAS** is solved numerically. Many numerical results are calculated, when the nonlocal term is neglected in the linear case and in the nonlinear case, and the error estimate, in each case, is computed. In addition, some special cases are derived and computed, in this work, when the memory takes different cases.

II. THE EXISTENCE OF UNIQUE SOLUTION

Throughout this section, the existence and uniqueness solution of Eq. (1.1) will be discussed and proved in the space $L_1[0,1]$, using Banach fixed point theorem. In this aim, we write Eq. (1.1) in the integral operator form:

$$\bar{U}(\phi(t)) = \frac{1}{\mu} f\left(t, H_1(\phi(t))\right) + \frac{1}{\mu}\, U(\phi(t)) \ , \tag{2.1}$$

where,

$$U(\phi(t)) = \lambda_1 \int_0^1 k(t,y) H_2(y,\phi(y)) dy + \lambda_2 \int_0^t F(t,\tau) H_3(\tau,\phi(\tau)) d\tau \ . \tag{2.2}$$

In addition, we consider the following conditions:

(i) The continuous kernel $k(t,y)$ satisfies: $|k(t,y)| \le M$, (M is a constant).

(ii) The continuous kernel $F(t,\tau)$, for all $t,\tau \in [0,T]$ satisfies:

$$|F(t,\tau)| \le V, (V \text{ is a constant }) .$$

(iii) The given functions $f\left(t, H_1(t,\phi(t))\right)$ and $H_s(t,\phi(t))$; $s = 2,3$ with their derivatives with respect to time t are continuous in the Banach space $L_1[0,1]$, $t < 1$, and satisfies for the constants p_1, p_2, p_3 and $p \ge max\{p_\ell\}; \ell = 1,2,3$, the following conditions:

$(a) \left|f\left(t, H_1(t,\phi(t))\right)\right| \le |f(t)| + p_1|\phi(t)|$,

where the norm of the function $f(t,0) = f(t)$ is defined by:

$$\|f(t)\|_{L_1[0,1]} = \int_0^1 |f(t)|\, dt \ = G \ , (G \text{ contant}) .$$

$(b) \left|H_s(t,\phi(t))\right| \le p_s|\phi(t)|$; $s = 2,3$.

Moreover, for the constants q_1, q_2, q_3 and $Q^* \ge \max\{p, q_\ell\}$; $\ell = 1,2,3$, we have for $\phi_1, \phi_2 \in L_1[0,1]$, the following conditions :

$(c) \left|f\left(t, H_1(t,\phi_1(t))\right) - f\left(t, H_1(t,\phi_2(t))\right)\right| \le q_1|\phi_1(t) - \phi_2(t)|$,

$(d) \left|H_s(t,\phi_1(t)) - H_s(t,\phi_2(t))\right| \le q_s|\phi_1(t) - \phi_2(t)|$; $s = 2,3$.

Theorem 1: If the conditions (i) – (iii) are satisfied, then Eq. (1.1) has a unique solution $\phi(t)$ in the space $L_1[0,1]$ under the condition:

$$Q^*(1 + |\lambda_1|M + |\lambda_2|TV) < \ |\mu|.$$

The proof of theorem 1 comes as a result of the following lemmas.

Lemma 1: If the conditions (i-iii-b) are verified, then the operator $\bar{U}$ defined by Eq. (2.1) maps the space $L_1[0,1]$ into itself.

Proof: In the light of the Eqs. (2.1) and (2.2), the normal form of the integral operator $\bar{U}$ is given by

$$\|\bar{U}\phi(t)\| \le \frac{1}{|\mu|}\left\{\left\|f\left(t, H_1(t,\phi(t))\right)\right\| + |\lambda_1|\left\|\int_0^1 |k(t,y)|\left|H_2(y,\phi(y))\right| dy\right\| + |\lambda_2|\left\|\int_0^t |F(t,\tau)|\left|H_3(\tau,\phi(\tau))\right| d\tau\right\|\right\}.$$

Using the conditions (i-iii-b), we get

$$\|\bar{U}\phi\| \le (G/|\mu|) + \sigma\|\phi\| \ , \ (\sigma = Q^*(1 + |\lambda_1|M + |\lambda_2|TV)/|\mu| , T = \max_{0\le t\le T} t). \tag{2.3}$$

The last inequality shows that, the operator $\bar{U}$ maps the ball $B_r \subset L_1[0,1]$ into itself, where

$$r = G/(|\mu| - Q^*(1 + |\lambda_1|M + |\lambda_2|TV)) \ . \tag{2.4}$$

Since $r > 0$, $G > 0$, therefore we have $\sigma < 1$. In addition, the inequality (2.2) involves the boundedness of the operator $\bar{U}$.

Lemma 2: If the conditions (i), (ii), (iii-c) and (iii-d) are verified, then the operator $\bar{U}$ defined by Eq. (2.1) is a contraction operator in the space $L_1[0,1]$.

Proof: For the two functions $\phi_1(t)$ and $\phi_2(t)$ in the space $L_1[0,1]$, the formulas (2.1) and (2.2) lead to

$$\begin{aligned}\|\bar{U}\phi_1(t) - \bar{U}\phi_2(t)\| &\\ \le \frac{1}{|\mu|}\Bigg\{&\left\|f\left(t, H_1(t,\phi_1(t))\right) - f\left(t, H_1(t,\phi_2(t))\right)\right\| \\ &+ |\lambda_1|\left\|\int_0^1 |k(t,y)|\left|H_2(y,\phi_1(y)) - H_2(y,\phi_2(y))\right| dy\right\| \\ &+ |\lambda_2|\left\|\int_0^t |F(t,\tau)|\left|H_3(\tau,\phi_1(\tau)) - H_3(\tau,\phi_2(\tau))\right| d\tau\right\|\Bigg\}.\end{aligned}$$

With the aid of conditions (i), (ii), (iii-c) and (iii-d), the above inequality becomes

$$\|\bar{U}\phi_1 - \bar{U}\phi_2\| \le \sigma\|\phi_1 - \phi_2\| \ , \quad \sigma = Q^*(1 + |\lambda_1|M + |\lambda_2|TV)/|\mu| \ . \tag{2.5}$$

Inequality (2.5) shows that, the operator $\bar{U}$ is continuous in the space $L_1[0,1]$ and then $\bar{U}$ is a contraction operator under the condition $\sigma < 1$.

Proof of theorem 1:

Lemmas 1 and 2 show that, the operator $\bar{U}$defined by Eq. (2.1) is a contraction operator in the Banach space $L_1[0,1]$. So, from Banach fixed point theorem, the operator $\bar{U}$ has a unique fixed point $\phi(t)$ which is the unique solution of Eq.(1.1).

III. COLLOCATION METHOD

In this section, **we** will use the Collocation method to obtain numerically the solution of Eq. (1.1).
For using the Collocation method, we approximate the solution $\phi(t)$ of Eq.(1.1) by the partial sum

$$Q_N(t)=\sum_{l=0}^{N} c_l\psi_l(t) \quad . \tag{3.1}$$

Where $\psi_0(t),\psi_1(t),\dots,\psi_N(t)$, are linearly independent functions defined in [0,1].
Using (3.1) in (1.1), we get

$$\mu\, Q_N(t)=f\Big(t,H_1\big(t,Q_N(t)\big)\Big)+\lambda_1\int_0^1 k(t,y)H_2\big(y,Q_N(y)\big)dy+\lambda_2\int_0^t F(t,\tau)H_3\big(\tau,Q_N(\tau)\big)d\tau$$
$$+E(t,c_1,c_2,\dots,c_N)\,. \tag{3.2}$$

Since the error E vanishes at $(N+1)$ points $t_0,t_1,t_2,\dots,t_N$; then after applying Trapezoidal rule the formula (3.2) can be adapted in the form

$$\mu\, Q_N(t_i)=f\Big(t_i,H_1\big(t_i,Q_N(t_i)\big)\Big)$$
$$+\lambda_1\sum_{j=0}^{N} u_j k(t_i,t_j)H_2\Big(t_j,Q_N(t_j)\Big)+\lambda_2\sum_{j=0}^{N} w_jF(t_i,t_j)H_3\Big(t_j,Q_N(t_j)\Big)+R_N' \quad . \tag{3.3}$$

Where R_N' is the error of the method; u_j and w_j are the weights defined by

$$u_j=\begin{cases} h/2 & ; \quad j=0,N \\ h & ; \quad 0<j<N \end{cases} \qquad , \qquad w_j=\begin{cases} h/2 & ; \quad j=0,i \\ h & ; \quad 0<j<i \\ 0 & ; \quad j>i \end{cases} \quad . \tag{3.4}$$

After neglecting the error R_N' and using formula (3.1), we obtain

$$\mu\sum_{l=0}^{N} c_l\psi_l(t_l)=f\left(t_i,H_1\left(\sum_{l=0}^{N} c_l\psi_l(t_i)\right)\right)+\lambda_1\sum_{j=0}^{N} u_jk(t_i,t_j)H_2\left(t_j,\sum_{l=0}^{N} c_l\psi_l(t_j)\right)$$
$$+\lambda_2\sum_{j=0}^{N} F(t_i,t_j)H_3\left(t_j,\sum_{l=0}^{N} c_l\psi_l(t_j)\right)\,. \tag{3.5}$$

Using the following notations, $\phi_i=\phi(t_i)$; $f_i\Big(H_{1,i}(\phi_i)\Big)=f\Big(t_i,H_1\big(t_i,\phi(t_i)\big)\Big)$;
$k(t_i,t_j)=k_{i,j}$; $F(t_i,t_j)=F_{i,j}$; $H_i(\phi_i)=H\big(t_i,\phi(t_i)\big)$, the formula (3.5) takes the form

$$\mu\sum_{l=0}^{N} c_l\psi_{l,i}=f_i\left(H_{1,l}\left(\sum_{l=0}^{N} c_l\psi_{l,i}\right)\right)+\lambda_1\sum_{j=0}^{N} u_jk_{i,j}H_{2,j}\left(\sum_{l=0}^{N} c_l\psi_{l,j}\right)$$
$$+\lambda_2\sum_{j=0}^{N} w_jF_{i,j}H_{3,j}\left(\sum_{l=0}^{N} c_l\psi_{l,j}\right)\,. \tag{3.6}$$

The formula (3.6) represents a nonlocal NAS of $(N+1)$ equations in $(N+1)$ unknowns $c_0,c_1,c_2,\dots,c_N$. By solving the system (3.6) , then substituting the unknowns $c_0,c_1,c_2,\dots,c_N$ in (3.1) , we get the approximation solution $Q_N(t)$ of Eq. (1.1).

Definition 1: The estimate local error R_N' of Eq.(3.3), for $0\le i\le N$, is determined by

$$R_N'=\left|\lambda_1\int_0^1 k(t,y)H_2\big(y,\phi(y)\big)dy+\lambda_2\int_0^t F(t,\tau)H_3\big(\tau,\phi(\tau)\big)d\tau-\lambda_1\sum_{j=0}^{N} u_jk_{i,j}H_{2,j}\left(\sum_{l=0}^{N} c_l\psi_{l,j}\right)\right.$$
$$\left.-\lambda_2\sum_{j=0}^{N} w_jF_{i,j}H_{3,j}\left(\sum_{l=0}^{N} c_l\psi_{l,j}\right)\right|$$
$$=-\frac{1}{12}h^2\frac{d^2}{d\xi^2}\left|k(t_N,\xi)H_2\left(\xi,\sum_{l=0}^{N} c_l\psi_l(\xi)\right)+F(t_N,\xi)H_3\left(\xi,\sum_{l=0}^{N} c_l\psi_l(\xi)\right)\right|;\xi\in(0,1)\,.$$

3.1. The existence and uniqueness solution of the nonlocal NAS:

In order to guarantee the existence of a unique solution of the nonlocal **NAS** (3.3) or (3.6) in the space ℓ^∞ , we consider the following conditions:

(1) The given sequences $\left\{f_i\Big(H_{1,i}(Q_{i,N})\Big)\right\}$, $\{H_{2,i}(Q_{i,N})\}$, $\{H_{3,i}(Q_{i,N})\}$ for all i , satisfy for the constants p_1',p_2',p_3' and $p'>max\{p_m'\}$; $m=1,2,3$, the following conditions :

(a') $\left|f_i\Big(H_{1,i}(Q_{i,N})\Big)\right|\le|f_i|+p_1'|Q_{i,N}|$,

Where , $\|f\|_{\ell^\infty}=\sup_i|f_i|=G'$, (G' *is a constant*).

$(b') \ \left|H_{s,i}(Q_{i,N})\right| \le p'_s \left|Q_{i,N}\right| \ ; \ s = 2,3$.
Moreover, for the constants q'_1, q'_2, q'_3 and $q' > max\{p', q'_m\}$; $m = 1,2,3$, we find
$(c') \ \left|f_i\left(H_{1,i}(Q_{i,N})\right) - f_i\left(H_{1,i}(Q'_{i,N})\right)\right| \le q_1 \left|Q_{i,N} - Q'_{i,N}\right|$.
$(d')\left|H_{s,i}(Q_{i,N}) - H_{s,i}(Q'_{i,N})\right| \le q_s \left|Q_{i,N} - Q'_{i,N}\right| \ ; s = 2,3$.
(2) $\sup_j \sum_{j=0}^{N} \left|u_j k_{i,j}\right| \le M'$, $\sup_j \sum_{j=0}^{N} \left|w_j F_{i,j}\right| \le V'$, $(M', V' are\ constants)$.
Theorem 2 (without proof): The nonlocal **NAS** (3.6) has a unique solution in Banach space ℓ^∞ under the condition : $q'(1 + |\lambda_1|M' + |\lambda_2|V') < |\mu|$.
Theorem 3.(without proof): The nonlocal **NAS** (3.6) has a unique solution in the Banach space ℓ^∞ under the condition : $q^*(1 + |\lambda_1|M^* + |\lambda_2|V^*)$.
Also for $N \to \infty$, the sum

$$\left\{\lambda_1 \sum_{j=0}^{N} u_j k(t_i, t_j) H_2\left(t_j, \sum_{l=0}^{N} c_l \psi_l(t_j)\right) + \lambda_2 \sum_{j=0}^{N} F(t_i, t_j) H_3\left(t_j, \sum_{l=0}^{N} c_l \psi_l(t_j)\right)\right\}$$
$$\to \left\{\lambda_1 \int_0^1 k(t,y) H_2\big(y, \varphi(y)\big) dy + \lambda_2 \int_0^t F(t,\tau) H_3\big(\tau, \varphi(\tau)\big) d\tau\right\}.$$

Thus, the solution of the nonlocal **NAS** (3.3) or (3.6) becomes the solution of the nonlocal **H-VIE** (1.1).
Theorem 4: If the sequence of continuous functions $\left\{f_N\left(t, H\big(t, \phi(t)\big)\right)\right\}$ converges uniformly to the function $f\left(t, H\big(t, \phi(t)\big)\right)$ in the space $L_1[0,1]$, then under the conditions (i), (ii), (iii-c) and (iii-d) of theorem (1) , the sequence of approximate solutions $\{\phi_N(t)\}$ converges uniformly to the exact solution of Eq. (1.1) in the space $L_1[0,1]$.
Proof: The formula (1.1) with its approximate solution gives

$$|\phi(t) - \phi_N(t)| \le \frac{1}{|\mu|}\left\{\left|f\left(t, H_1\big(t, \phi(t)\big)\right) - f_N\left(t, H_1\big(t, \phi(t)\big)\right)\right| + |\lambda_1| \int_0^1 |k(t,y)| \left|H_2\big(y, \phi(y)\big) - H_2\big(y, \phi_N(y)\big)\right| dy \right.$$
$$\left. + |\lambda_2| \int_0^t |F(t,\tau)| \left|H_3\big(\tau, \phi(\tau)\big) - H_3\big(\tau, \phi_N(\tau)\big)\right| d\tau \right\}.$$

Using the conditions (i), (ii), (iii-c) and (iii-d), of theorem (1), we get

$$\|\phi(t) - \phi_N(t)\|_{L_1[0,1]}$$
$$\le \frac{1}{\big(|\mu| - Q^*(|\lambda_1|M + |\lambda_2|VT)\big)} \left\|f\left(t, H_1\big(t, \phi(t)\big)\right) - f_N\left(t, H_1\big(t, \phi(t)\big)\right)\right\|_{L_1[0,1]}.$$

Hence **,** $\|\phi(t) - \phi_N(t)\|_{L_1[0,1]} \to 0$ since $\left\|f\left(t, H_1\big(t, \phi(t)\big)\right) - f_N\left(t, H_1\big(t, \phi(t)\big)\right)\right\|_{L_1[0,1]} \to 0$ as $N \to \infty$ **.**
Corollary 1: The total error R_N satisfies $\lim_{N\to\infty} R_N = 0$.
Proof: From the definition of the R_N, we have

$$R_N = [\phi_i - (\phi_i)_N] - \left[\lambda_1 \sum_{j=0}^{N} u_j k_{i,j} \big(H_{2,j}(\phi_i) - H_{2,j}(\phi_i)_N\big) + \lambda_2 \sum_{j=0}^{N} w_j\ F_{i,j} \big(H_{3,j}(\phi_i) - H_{3,j}(\phi_i)_N\big)\right]$$

The above formula can be adapted in the form

$$|R_N| \le \sup_i |\phi_i - (\phi_i)_N| + |\lambda_1| \sup_j \sum_{j=0}^{N} \left|u_j k_{i,j}\right| \ \sup_j \left|\big(H_{2,j}(\phi_i) - H_{2,j}(\phi_i)_N\big)\right|$$
$$+ |\lambda_2| \sup_j \sum_{j=0}^{N} \left|w_j\ F_{i,j}\right| \ \sup_j \left|H_{3,j}(\phi_i) - H_{3,j}(\phi_i)_N\right| .$$

With the aid of conditions (1-d') and (2), the above inequality becomes
$|R_N| \le (1 + |\lambda_1|M'q' + |\lambda_2|V'q')\|\phi(t_i) - \phi_N(t_i)\|_{\ell^\infty}$, $\forall N$.
Since each term R_N is bounded above , hence for $t = t_i$, we deduce

$$\|R_N\|_{\ell^\infty} \le (1 + |\lambda_1|M'q' + |\lambda_2|V'q')\|\phi(t) - \phi_N(t)\|_{L_1[0,1]} .$$

Since $\|\phi(t) - \phi_N(t)\|_{L_1[0,1]} \to 0$ as $N \to \infty$, then $\lim_{N\to\infty}\|R_N\|_{\ell^\infty} = 0$.

IV. NUMERICAL EXAMPLES

For the nonlocal **H-VIE:**

$$\mu\, \phi(t) = f\left(t, H_1\big(t, \phi(t)\big)\right) + 0.01 \int_0^1 t y^2 H_2\big(y, \phi(y)\big) dy + 0.01 \int_0^t t\tau H_3\big(\tau, \phi(\tau)\big) d\tau . \quad (4.1)$$

$(\phi(t) = t^2, \lambda_1 = \lambda_2 = 0.01 ,\ 0 \le t \le T < 1)$.

We use the Collocation method to obtain the numerical solution for the nonlocal **H-VI** (4.1) for different value of ($h = 0.25\,, 0.125\,, 0{,}625$) and μ for several forms of $H_1\big(t,\phi(t)\big)$; $H_2\big(t,\phi(t)\big)$; $H_3\big(t,\phi(t)\big)$ as the following:

Case (I) when there is no memory term ($H_1\big(t,\phi(t)\big) = 0$).

Here we solve, numerically (4.1) for different value of $\mu = (0.1\,, 0.5\,, 1)$ and $h = 0.625$.

(Case I.1) when there is no memory term $(H_1(t,\phi(t) = 0))$ and for the nonlinear functions $H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$**.**

Table (1)

Case I.1: collocation method $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$							
t	ϕ	$\mu = 0.1\,, h = 0.625\,, N = 16$		$\mu = 0.5\,, h = 0.625\,, N = 16$		$\mu = 1\,, h = 0.625\,, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25596E-02	5.96000E-05	6.25115E-02	1.15000E-05	6.25057E-02	5.72547E-06
0.5	2.50000E-01	2.50124E-01	1.24000E-04	2.50024E-01	2.40000E-05	2.50012E-01	1.20000E-05
0.75	5.62500E-01	5.62719E-01	2.19000E-04	5.62542E-01	4.20000E-05	5.62521E-01	2.10000E-05
1	1.00000E+00	1.00041E+00	4.10000E-04	1.00008E+00	8.00000E-05	1.00004E+00	4.00000E-05

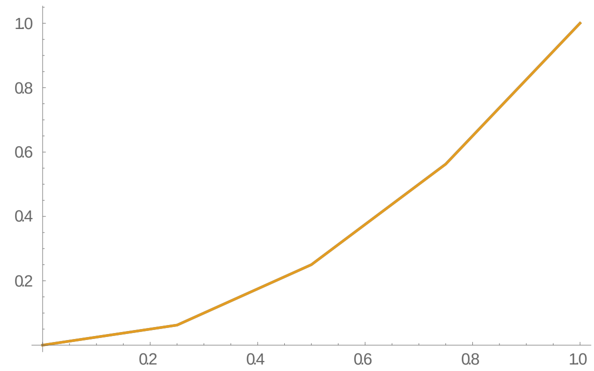

Fig.(1-i) $\mu = 0.1\,, h = 0.625$

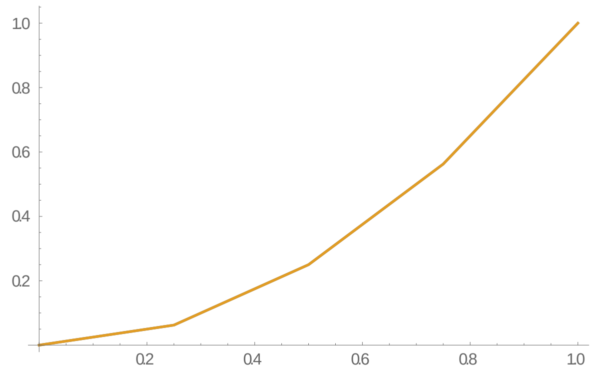

Fig. (1-ii) $\mu = 0.5\,, h = 0.625$

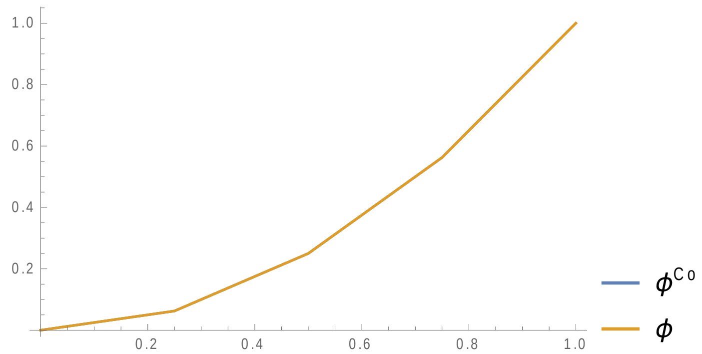

Fig.(1-iii) $\mu = 1\,, h = 0.625$

Figs. (1) describe the relation between the exact solution and numerical solution, when $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$, using **Collocation method**, with $(\lambda = 0.01\,, h = 0.625\ and\ N = 16)$ at $\mu = 0.1$ in Fig. (1.i), $\mu = 0.5$ in Fig (1.ii) and $\mu = 1$ in Fig. (1.iii).

Table (2)

Case I.2: Collocation method $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi(t)$							
t	ϕ	$\mu = 0.1\,, h = 0.625\,, N = 16$		$\mu = 0.5\,, h = 0.625\,, N = 16$		$\mu = 1\,, h = 0.625\,, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25607E-02	6.07000E-05	6.25118E-02	1.18000E-05	6.25059E-02	5.90000E-06
0.5	2.50000E-01	2.50131E-01	1.31000E-04	2.50025E-01	2.50000E-05	2.50013E-01	1.30000E-05
0.75	5.62500E-01	5.62722E-01	2.22000E-04	5.62543E-01	4.30000E-05	5.62521E-01	2.10000E-05
1	1.00000E+00	1.00034E+00	3.40000E-04	1.00007E+00	7.00000E-05	1.00003E+00	3.00000E-05

Table (2) describes the relation between the exact solution and numerical solution, when $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi(t)$, using **Collocation method**, with $(\lambda = 0.01\,, h = 0.625\ \ and\ N{=}16$ at $\mu{=}0.1$,0.5,1.

Table (3)

Case I.3: Collocation method $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = \phi(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$							
t	ϕ	$\mu = 0.1\,, h = 0.625\,, N = 16$		$\mu = 0.5\,, h = 0.625\,, N = 16$		$\mu = 1\,, h = 0.625\,, N = 16$	
		ϕ^{C_o}	E^{C_o}	ϕ^{C_o}	E^{C_o}	ϕ^{C_o}	E^{C_o}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25341E-02	3.40901E-05	6.25066E-02	6.59313E-06	6.25033E-02	3.28304E-06
0.5	2.50000E-01	2.50073E-01	7.30247E-05	2.50014E-01	1.41410E-05	2.50007E-01	7.04258E-06
0.75	5.62500E-01	5.62642E-01	1.41560E-04	5.62527E-01	2.74454E-05	5.62514E-01	1.36706E-05
1	1.00000E+00	1.00031E+00	3.07695E-04	1.00006E+00	5.91340E-05	1.00003E+00	2.94224E-05

Table (3) describe the exact solution and numerical solution, when $H_1\big(t,\phi(t)\big) = 0\,, H_2\big(t,\phi(t)\big) = \phi(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$, using **Collocation method**, with $(\lambda = 0.01\,, h = 0.625\ \ and\ N = 16)$ at $\mu = (0.1{,}0.5{,}1)$.

Case (II) When $H_1\big(t,\phi(t)\big)$ takes a linear form $(H_1\big(t,\phi(t)\big) = t\phi(t))$.

Table (4)

Case II.1: collocation method $H_1\big(t,\phi(t)\big) = t\phi(t)\,, H_2\big(t,\phi(t)\big) = t\phi^2(t)\,, H_3\big(t,\phi(t)\big) = \phi^2(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_o}	E^{C_o}	ϕ^{C_o}	E^{C_o}	ϕ^{C_o}	E^{C_o}
0	0.00000E+00	5.56156E-31	5.56156E-31	1.84817E-27	1.84817E-27	-2.89127E-19	-2.89127E-19
0.25	6.25000E-02	6.28549E-02	3.54900E-04	6.25908E-02	9.08000E-05	6.25228E-02	2.28000E-05
0.5	2.50000E-01	2.50370E-01	3.70000E-04	2.50095E-01	9.50000E-05	2.50024E-01	2.40000E-05
0.75	5.62500E-01	5.62936E-01	4.36000E-04	5.62611E-01	1.11000E-04	5.62528E-01	2.80000E-05
1	1.00000E+00	1.00061E+00	6.10000E-04	1.00016E+00	1.60000E-04	1.00004E+00	4.00000E-05

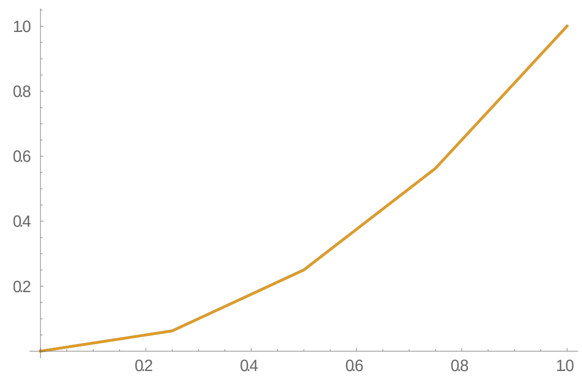

Fig.(4-i) $h = 0.25, N = 4$

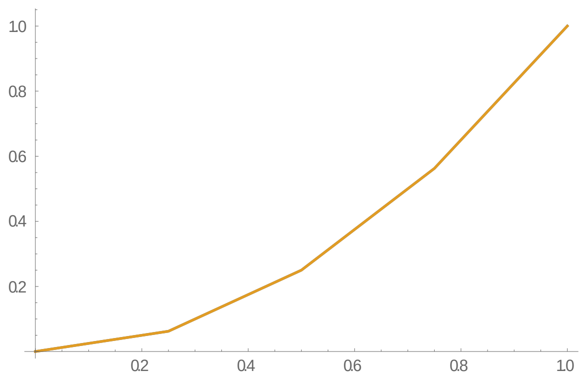

Fig.(4-ii) $h = 0.125, N = 8$

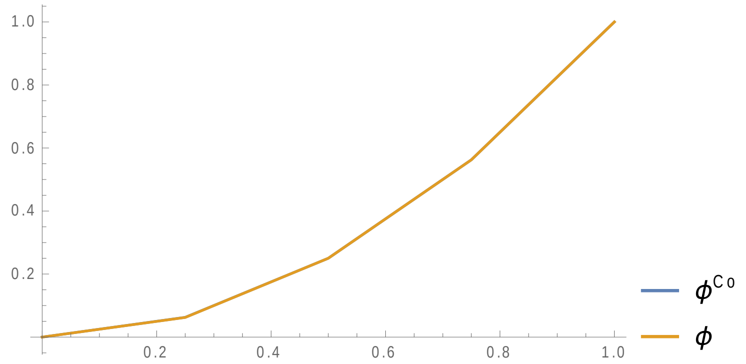

Fig.(4-iii) $h = 0.625, N = 16$

Figs. (4) describe the relation between the exact solution and numerical solution, when $H_1(t,\phi(t)) = t\phi(t)\,, H_2(t,\phi(t)) = t\phi^2(t)\,, H_3(t,\phi(t)) = \phi^2(t)$, with $\lambda = 0.01\,, \mu = 0.001$ at $h = 0.25, N = 4\,;\ h = 0.125, N = 8\,;\ h = 0.625, N = 16$ in Fig. (4.i), Fig. (4.ii) and Fig.(4.iii), respectively.

Table (5)

CaseII.2 Collocation method $H_1(t,\phi(t)) = t\,\phi(t)\,, H_2(t,\phi(t)) = t\phi^2(t)\,, H_3(t,\phi(t)) = \phi(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	5.90505E-31	5.90505E-31	-4.35047E-27	-4.35047E-27	2.30194E-18	2.30194E-18
0.25	6.25000E-02	6.28637E-02	3.63700E-04	6.25930E-02	9.30000E-05	6.25234E-02	2.34000E-05
0.5	2.50000E-01	2.50394E-01	3.94000E-04	2.50101E-01	1.01000E-04	2.50025E-01	2.50000E-05
0.75	5.62500E-01	5.62944E-01	4.44000E-04	5.62613E-01	1.13000E-04	5.62528E-01	2.80000E-05
1	1.00000E+00	1.00051E+00	5.10000E-04	1.00013E+00	1.30000E-04	1.00003E+00	3.00000E-05

Table (5) describe the exact solution and numerical solution, when $H_1(t,\phi(t)) = t\,\phi(t)\,, H_2(t,\phi(t)) = t\phi^2(t)\,, H_3(t,\phi(t)) = \phi(t)$, $\lambda_{1,2} = 0.01$ at $h = 0.25,\ N = 4\,;\ h = 0.125,\ N = 8\,;\ h = 0.625,\ N = 16$.

Table (6)

Case II.3: Collocation method $H_1(t,\phi(t)) = t\phi(t)\,, H_2(t,\phi(t)) = \phi(t)\,, H_3(t,\phi(t)) = \phi^2(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	2.30450E-30	2.30450E-30	-1.13980E-28	-1.13980E-28	-2.31631E-18	-2.31631E-18
0.25	6.25000E-02	6.27082E-02	2.08182E-04	6.25523E-02	5.23091E-05	6.25131E-02	1.30937E-05
0.5	2.50000E-01	2.50223E-01	2.23295E-04	2.50056E-01	5.61993E-05	2.50014E-01	1.40733E-05
0.75	5.62500E-01	5.62789E-01	2.88835E-04	5.62573E-01	7.27665E-05	5.62518E-01	1.82264E-05
1	1.00000E+00	1.00047E+00	4.66820E-04	1.00012E+00	1.17498E-04	1.00003E+00	2.94241E-05

Table (6) describe the exact and numerical solution, when $H_1(t,\phi(t)) = t\phi(t)\,,\ H_2(t,\phi(t)) = \phi(t)\,,\ H_3(t,\phi(t)) = \phi^2(t)$, with $\lambda_{1,2} = 0.01\,, \mu = 0.001$ at $h = 0.25,\ N = 4\,;\ h = 0.125,\ N = 8\,;\ h = 0.625,\ N = 16$.

Case (III) When $H_1(t,\phi(t))$ takes a nonlinear form ($H_1(t,\phi(t)) = \phi^2(t)$).

Table (7)

Case III.1: Collocation method $H_1(t,\phi(t)) = \phi^2(t)\,, H_2(t,\phi(t)) = t\phi^2(t)\,, H_3(t,\phi(t)) = \phi^2(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	1.22705E-29	1.22705E-29	4.90379E-29	4.90379E-29	6.34499E-17	6.34499E-17
0.25	6.25000E-02	6.32013E-02	7.01269E-04	6.26802E-02	1.80197E-04	6.25454E-02	4.53616E-05
0.5	2.50000E-01	2.50369E-01	3.69187E-04	2.50095E-01	9.45529E-05	2.50024E-01	2.37816E-05
0.75	5.62500E-01	5.62790E-01	2.90150E-04	5.62574E-01	7.41490E-05	5.62519E-01	1.86394E-05
1	1.00000E+00	1.00031E+00	3.06496E-04	1.00008E+00	7.79606E-05	1.00002E+00	1.95748E-05

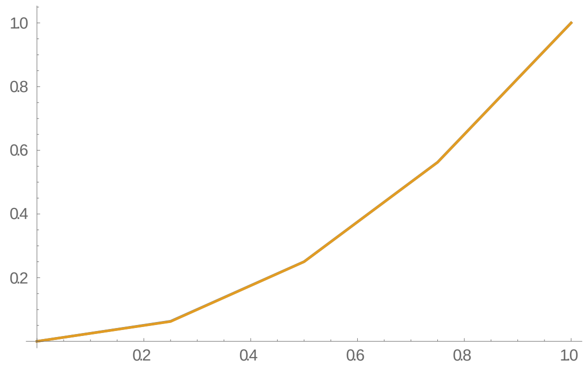

Fig.(7-i) $h = 0.25, N = 4$

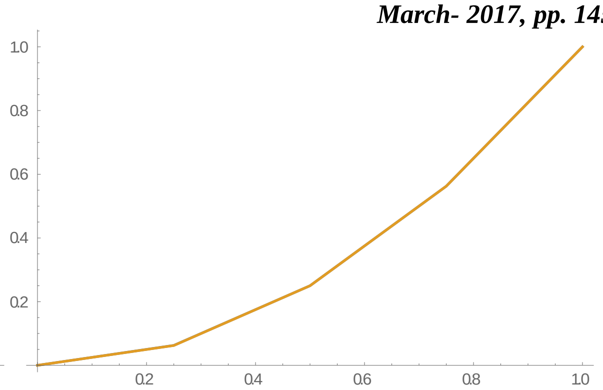

Fig. (7-ii) $h = 0.125, N = 8$

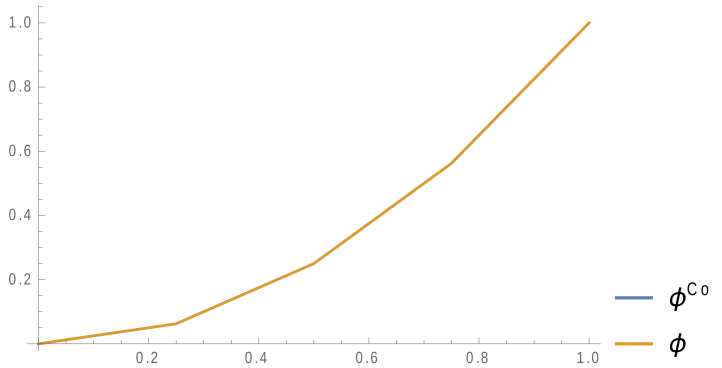

Fig.(7-iii) $h = 0.625, N = 16$

Figs. (7) describe the exact and numerical solution, when $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi^2(t)$, for different values of h and N with $\lambda_{1,2} = 0.01$, $\mu = 0.01$ in Fig. (7.i), Fig (7.ii) and Fig. (7.iii), respectively.

Table (8)

Case III.2: Collocation method $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	-1.08517E-31	1.08517E-31	7.38095E-27	7.38095E-27	2.44597E-16	2.44597E-16
0.25	6.25000E-02	6.32190E-02	7.18992E-04	6.26847E-02	1.84665E-04	6.25465E-02	4.64838E-05
0.5	2.50000E-01	2.50393E-01	3.93112E-04	2.50100E-01	1.00431E-04	2.50025E-01	2.52460E-05
0.75	5.62500E-01	5.62795E-01	2.95365E-04	5.62575E-01	7.52560E-05	5.62519E-01	1.89048E-05
1	1.00000E+00	1.00026E+00	2.56096E-04	1.00007E+00	6.50819E-05	1.00002E+00	1.63384E-05

Table (8) describe the exact and numerical solution, when $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi(t)$, for different values of h and N, at $\lambda_{1,2} = \mu = 0.01$.

Table(9)

Case III.3: Collocation method $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = \phi(t)$, $H_3(t,\phi(t)) = \phi^2(t)$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.625, N = 16$	
		ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}	ϕ^{C_0}	E^{C_0}
0	0.00000E+00	1.27105E-31	1.27105E-31	9.25830E-27	9.25830E-27	1.21226E-16	1.21226E-16
0.25	6.25000E-02	6.29126E-02	4.12590E-04	6.26039E-02	1.03948E-04	6.25260E-02	2.60373E-05
0.5	2.50000E-01	2.50223E-01	2.22816E-04	2.50056E-01	5.61112E-05	2.50014E-01	1.40532E-05
0.75	5.62500E-01	5.62692E-01	1.92288E-04	5.62548E-01	4.84622E-05	5.62512E-01	1.21399E-05
1	1.00000E+00	1.00023E+00	2.32915E-04	1.00006E+00	5.86500E-05	1.00001E+00	1.46888E-05

Table (9) describe the exact and numerical solution, when $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = \phi(t)$, $H_3(t,\phi(t)) = \phi^2(t)$, and for different values of h and N with $\lambda_{1,2} = \mu = 0.01$.

V. CONCLUSIONS

From the above results and others results we obtained, we can see that the proposed method is efficient and accurate, also we note the following:

1. The value of absolute error is decreasing when the value of h decreases for all cases of studies.
2. The absolute value of the error when the memory term $H_1(t,\phi(t))$ takes a nonlinear form is less than the corresponding error of the linear several forms of $H_2(t,\phi(t))$; $H_3(t,\phi(t))$.
3. When the memory term $H_1(t,\phi(t)) = 0$, the absolute value of the error is large when $\mu \leq 0.001$ ($\mu \ll 1$) for several forms of $H_2(t,\phi(t))$; $H_3(t,\phi(t))$.
4. The value of absolute error is decreasing when the value of μ increases when the memory term $H_1(t,\phi(t)) = 0$, in the several forms of $H_2(t,\phi(t))$; $H_3(t,\phi(t))$.
5. In the nonlocal integral equations μ called the phase-lag of the integral equations.
6. The Max. E. and Min. E. in all cases of studies are given as follow:

(6.1). For the case $H_1(t,\phi(t)) = 0$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi^2(t)$; we have Max. E. and Min. E. in Table (1) at ($h = 0.625, \lambda = 0.01$) as follow:
when $\mu = 0.01$: (at $t = 1$) 4.10000E-04 and (at $t = 0$) 0.00000E+00, respectively .
when $\mu = 0.5$: (at $t = 1$) 8.00000E-05 and (at $t = 0$) 0.00000E+00, respectively.
when $\mu = 1$: (at $t = 1$) 4.00000E-05 and (at $t = 0$) 0.00000E+00, respectively.
(6.2) For the case $H_1(t,\phi(t)) = t\phi(t)$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi^2(t)$; we have we have Max. E. and Min. E. in Table (4) at ($h = 0.625, \lambda = 0.01$) as follow:
when $h = 0.25$: (at $t = 1$) 6.10000E-04 and (at $t = 0$) 5.56156E-31, respectively.
when $h = 0.125$: (at $t = 1$) 1.60000E-04 and (at $t = 0$) 1.84817E-27, respectively.
when $h = 0.625$: (at $t = 1$) 4.00000E-05 and (at $t = 0$) 2.89127E-19, respectively.
(6.3) For the case $H_1(t,\phi(t)) = \phi^2(t)$, $H_2(t,\phi(t)) = t\phi^2(t)$, $H_3(t,\phi(t)) = \phi^2(t)$; we have we have Max. E. and Min. E. in Table (7) at ($h = 0.625, \lambda = 0.01$) as follow:
when $h = 0.25$: (at $t = 0.25$) 7.01269E-04 and (at $t = 0$) 1.22705E-29, respectively.
when $h = 0.125$: (at $t = 0.25$) 1.80197E-04 and (at $t = 0$) 4.90379E-29, respectively.
when $h = 0.625$:(at $t = 0.25$) 4.53616E-05 and (at $t = 0$) 6.34499E-17, respectively.

REFERENCES

[1] C. D. Green, *Integral Equation Methods*, Nelson, New York, 1969.
[2] H. Hochstadt, *Integral Equations*, John Wiley, New York, 1973.
[3] R. P. Kanwal, *Linear Integral Equations Theory and Technique*, Boston, 1996.
[4] P. Schiavone, C.Constanda and A. Mioduchowski , *Integral Methods in Science and Engineering*, Birkhauser Boston, 2002.
[5] M. A. Abdou, On asymptotic in contact problems of Fredholm equation of the second kind, *Korean J. Appl. Math. Comp.* Vol. 9 No. 1 (2002) 261-276
[6] M. A. Abdou, Fredholm – Volterra equation of the first kind and contact problem, *Appl. Math. Compute*. 125 (2002) 177- 193
[7] Z. Z. Bai, J.-F. Yin and Y.-F. Su, A shift-splitting preconditioner for non-Hermitian positive definite matrices, *J. Comput. Math.*, 24 (2006) 539-552.
[8] A. Tari, M. Rahimi, S. Shahmorad, F. Talati, Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method, *J. Comp. Appl. Math.* 228 (2009) 70-76.
[9] T. A. Burton, *Volterra Integral and Differential Equations*, London, New York, 1983.
[10] Peter Linz, *Analytic and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.
[11] N.I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen, the Netherland, 1953.
[12] V.M. Aleksandrov and E.V. Covalence, *Problems in the Mechanics of Continuous Media and Mixed Boundary Conditions*, Nauka Moscow, 1986.
[13] M. H. Reihani, Z. Abadi, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, *J. Comp. Appl. Math.* 200 (2007) 12-20.
[14] A. Cardone, E. Messina, E. Russo, A fast iterative method for discretized Volterra Fredholm integral equations, *J. Comp. App. Math.* 189 (2006) 568-579.
[15] M. A. Golberg. ed, *Numerical Solutions for Integral Equations*, New York, 1990.
[16] L. M. Delves, J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge, 2011.
[17] K. E. Atkinson, *A Survey of Numerical Method for the Solution of Fredholm Integral Equation of the Second Kind*, Philadelphia, 1976.
[18] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge, 2011.
[19] T. H. Baker, *Treatment of Integral Equations by Numerical Methods*, Academic Press, 1982.

[20] M. A. Abdou, M. M. EL-Borai, M. M. El-Kojok, Toeplitz matrix method and nonlinear integral equation of Hammerstein type, *J. Comp. Appl. Math.* Vol. 223, (2009) 765 – 776

[21] S. Bazm, E. Babolian, Numerical solution of nonlinear two-dimensional fredholm integral equations of the second kind using gauss product quadrature rules, *Commun. Nonlinear Sci. Numer. Simult.* 17 (2012) 1215-1223.

[22] D. Liang, P. Weng, J. Wu, Traveling wave solutions in a delayed predator-prey diffusion PDE system: point-to-periodic and point-to point waves, *IMA J. Appl. Math.* 77 (2012) 516-545.

[23] Y. Cao, L.-Q. Yao, M.-Q. Jiang and Q. Niu, A relaxed HSS preconditioned for saddle point problems from mesh free discretization, *J. Compute. Math.*, 21 (2013) 398-421.

[24] I. Aziz, Siraj-ul-Islam, New algorithms for numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, *J. Comp. Appl. Math.* 239 (2013) 333-345.

[25] P. Assari, H. Adibi, M. Dehghal, A meshless method for solving nonlinear two-dimensional integral equations of the second kind on non-rectangular domains using radial basis functions with error analysis, *J. Comp. Appl. Math.* 239 (2013) 72-92.