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On Simultaneous Approximation of Functions by a Certain Family of Linear Positive Operators

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***Abstract:** In the present paper we introduce a new family of linear positive operators and study some direct results in simultaneous approximation. The aim of the present paper is to establish pointwise convergence formula, a Voronovskaja-type asymptotic formula and an error estimation formula in terms of modulus of continuity, in simultaneous approximation.*

***Key Words and Phrases:** Simultaneous approximation, Linear positive operators, Modulus of continuity, Voronovskaja-type asymptotic formula, pointwise convergence.*

I. INTRODUCTION

Recently Gal and Gupta [2], obtained approximation properties of a complex Szasz-Durrmeyer operator in a compact disks. Very recently Gupta [3], studied direct estimates for a new general family of Durrmeyer type operators. In the present paper we introduce a new family of linear positive operators and study some direct results in simultaneous approximation [4, 5,6,7, 8].

For $f \in C_\gamma[0,\infty) \equiv \{f \in C[0,\infty) : f(t) \le Mt^\gamma$ for some $M > 0,\ \gamma > 0\}$, we define a new sequence of summation integral type operators

$$B_n(f,x) = \frac{n}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty s_{n,v}(t) f(t)dt \qquad x \in [0,\infty) \tag{1}$$

where

$$b_{n,v}(x) = \frac{x^{v-1}}{B(n+1,v)(1+x)^{n+v+1}}, \quad s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!} \qquad \text{and}$$
$$B(n+1,v) = n!(v-1)!/(n+v)!.$$

It can be is easily checked that the operators B_n defined by (1) are linear positive operators [9] and it is obvious that $B_n(1,x) = 1$. Alternately the operators (1) may be written as

$$B_n(f,x) = \int_0^\infty W_n(x,t) f(t)dt, \qquad \text{where} \qquad W_n(x,t) = \frac{n}{n+1}\sum_{k=1}^{\infty} b_{n,v}(x)s_{n,v}(t)$$

As far as the rate of approximation is concerned the operators (1) are just like exponential type operators [10]. But the operators B_n are not exponential type operators, since they do not satisfy the following condition:

$$\frac{\partial}{\partial x}W_n(x,t) = \frac{n}{P(x)}W_n(x,t)(t-x), \tag{2}$$

where $P(x)$ is a function of *x*.

The above equation (2) is the necessary condition for any operator to be of exponential type. The above condition (2) is frequently used in the analysis to prove the inverse theorem [1, 10] for exponential type operators. In our case we have tried to overcome this difficulty and here in the present paper we obtain a point wise convergence, a Voronovskaja type asymptotic formula and an error estimation formula in simultaneous approximation for the operators B_n , defined by (1).

II. AUXILIARY RESULTS

To prove the main results we need the following auxiliary results.

Lemma 2.1 [4]. For $m \in N \cup \{0\}$, if the m^{th} order moment be defined as

$$U_{n,m}(x)=\frac{1}{n+1}\sum_{k=1}^{\infty}b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^m,$$

then we have $U_{n,0}(x)=1$, $U_{n,1}(x)=0$ and

$$(n+2)U_{n,m+1}(x)=x(1+x)[U_{n,m}^{(1)}(x)+mU_{n,m-1}(x)].$$

Consequently

$$U_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right).$$

Lemma 2.2. Let the function $\mu_{n,m}(x), m\in N^0$, be defined as

$$\mu_{n,m}(x)=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^m\,dt.$$

Then $\mu_{n,0}(x)=1, \mu_{n,1}(x)=\dfrac{2(x+1)}{n}$ and $\mu_{n,2}(x)=\dfrac{x(x+2)n+6(1+x)^2}{n^2}$

and there holds the following recurrence relation

$$n\,\mu_{n,m+1}(x)=x(1+x)[\mu_{n,m}^{(1)}(x)+m\mu_{n,m-1}(x)]+mx\mu_{n,m-1}(x)+[m+2(x+1)]\mu_{n,m}(x).$$

Consequently for each $x\in[0,\infty)$ we have from this recurrence relation that

$$\mu_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right).$$

Proof. The values of $\mu_{n,0}(x), \mu_{n,1}(x)$ easily follow from the definition. We prove the recurrence relation

$$x(1+x)\mu_{n,m}^{(1)}(x)=\frac{n}{n+1}\sum_{v=1}^{\infty}x(1+x)b_{n,k}^{(1)}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^m\,dt$$

$$-\frac{mn}{n+1}\sum_{v=1}^{\infty}x(1+x)b_{n,k}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^{m-1}dt$$

Now using the identities $x(1+x)b_{n,v}^{(1)}(x)=((v-1)-(n+2)x)b_{n,v}(x)$ and $t.s_{n,v}^{(1)}(t)=[(v-nt]s_{n,v}(t)$, we obtain

$$x(1+x)[\mu_{n,m}^{(1)}(x)+m\mu_{n,m-1}(x)]$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}(v-1-(n+2)x)b_{n,v}(x)\int_0^{\infty}s_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}[(v-nt)+n(t-x)-(1+2x)]s_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}t\,s_{n,v}^{(1)}(t)\,(t-x)^m\,dt\;+\;n\,\mu_{n,\,m+1}(x)-(1+2x)\mu_{n,m}(x)$$

$$=\frac{n}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}s_{n,v}^{(1)}(t)\,(t-x)^{m+1}dt\;+\;\frac{nx}{n+1}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}s_{n,v}^{(1)}(t)\,(t-x)^m\,dt$$

$$+n\,\mu_{n,\,m+1}(x)-(1+2x)\mu_{n,m}(x).$$

$$=-(m+1)\mu_{n,m}(x)+n\mu_{n,m+1}(x)-mx\mu_{n,m-1}(x)-(1+2x)\,\mu_{n,m}(x).$$

This completes the proof of recurrence relation. The values of $\mu_{n,2}(x)$, $\mu_{n,m}(x)$ follow from the recurrence relation.

Corollary 2.3. Let δ be a positive number. Then for every $\gamma>0, x\in(0,\infty)$, there exists a constant *M(s, x)* independent of n and depending on s and x such that

$$\int_{|t-x|>\delta}W_n(x,t)e^{\gamma t}dt\le M(s,x)n^{-s},\quad \text{where}\quad s=1,\,2,\,3,\,.....$$

Corollary 2.4. For the operators $B_n(f,x)$ we have the following result:

$$B_n(t,x)=\frac{2+n}{n}x+\frac{2}{n} \quad B_n(t^2,x)=\frac{(n+2)(n+3)}{n^2}x^2+\frac{6(n+2)}{n^2}x+\frac{6}{n^2}$$

and in general

$$B_n(t^i,x)=\frac{(n+i+1)!}{(n+1)!n^i}x^i+\frac{i(i+1)\times(n+i)!}{(n+1)!n^i}x^{i-1}+O(n^{-2}),\ i\geq 2$$

Lemma 2.4 [4]. There exist the polynomials $Q_{i,j,r}(x)$ independent of n and k such that

$$\{x(1+x)\}^r D^r[b_{n,v}(x)]=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+2)^i(v-1-(n+2)x)^j Q_{i,j,r}(x)b_{n,v}(x),\ \text{where } D\equiv\frac{d}{dx}.$$

III. DIRECT RESULTS

The following theorem is the point wise convergence in simultaneous approximation for the operators (1).

Theorem 3.1 Let $f\in C_\gamma[0,\infty),\gamma>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then

$$B_n^{(r)}(f(t),x)=f^{(r)}(x)+o(1) \text{ as } n\to\infty. \qquad (3)$$

Further if $f^{(r)}$ exists and is continuous on $(a-\eta,b-\eta)\subset(0,\infty)$, $\eta>0$, then (3) holds uniformly in $x\in[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x)\to 0$ as $t\to\infty$.

Now

$$B_n^{(r)}(f,x)=\int_0^\infty W_n^{(r)}(t,x)f(t)dt$$

$$=\sum_{i=0}^r\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt+\int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt$$

$$=R_1+R_2 \qquad \text{(say).}$$

First to estimate R_1, using binomial expansion of $(t-x)^m$ and Corollary 2.4, we have

$$R_1=\sum_{i=0}^r\frac{f^{(i)}(x)}{i!}\sum_{v=0}^i\binom{i}{v}(-x)^{i-v}\frac{\partial^r}{\partial x^r}\int_0^\infty W_n(t,x)t^v dt$$

Next using Lemma 2.5, we obtain

$$R_2=\int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt$$

$$=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+2)^i\frac{Q_{i,j,r}(x)}{\{x(1+x)\}^r}\sum_{v=1}^\infty\{v-1-(n+2)x\}^j b_{n,v}(x)\int_0^\infty s_{n,v}(t)\varepsilon(t,x)(t-x)^r dt+$$

$$(-1)^r\frac{(n+r-1)!}{n!}\varepsilon(0,x)(-x)^r$$

$$=R_3+R_4 \qquad \text{(say).}$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$ for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. Thus for some $M_1>0$, we can write

$$|R_3|\leq M_1\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+2)^i\sum_{v=1}^\infty b_{n,v}(x)|v-1-(n+2)x|^j\left\{\varepsilon\int_{|t-x|<\delta}s_{n,v}(t)|t-x|^r\int_{|t-x|\geq\delta}s_{n,v}(t)M_2t^\gamma dt\right\}$$

$$=R_5+R_6,\ \text{say,}$$

where $M_1 = \sup_{\substack{2i+j\le r \\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{\{x(1+x)\}^r}$ and M_2 is independent of t.

Applying Schwarz inequality for integration and summation respectively, we obtain

$$R_5 \le \varepsilon M_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i \sum_{v=1}^{\infty} b_{n,v}(x)|v-1-(n+2)x|^j \left(\int_0^\infty s_{n,v}(t)dt\right)^{1/2} \left(\int_0^\infty s_{n,v}(t)(t-x)^{2r}dt\right)^{1/2}$$

$$\le \varepsilon M_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i \left(\sum_{v=1}^{\infty} b_{n.v}(x)(v-1-(n+2)x)^{2j}\right)^{1/2} \left(\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty s_{n,v}(t)(t-x)^{2r}dt\right)^{1/2}$$

Using Lemma 2.1 and Lemma 2.2, we get

$$R_5 \le \varepsilon M_1 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i\, O\left(n^{j/2}\right) O\left(n^{-r/2}\right) = \varepsilon\, O(1)$$

Again using Schwarz inequality, Lemma 2.1 and Corollary 2.3, we get

$$R_6 \le M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i \sum_{v=1}^{\infty} b_{n,v}(x)|v-1-(n+2)x|^j \int_{|t-x|\ge\delta} s_{n,v}(t)\, t^{\gamma} dt$$

$$\le M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i \sum_{v=1}^{\infty} b_{n,v}(x)|v-1-(n+2)x|^j \left(\int_{|t-x|\ge\delta} s_{n,v}(t)dt\right)^{1/2} \left(\int_{|t-x|\ge\delta} s_{n,v}(t)t^{2\gamma}dt\right)^{1/2}$$

$$\le M_2 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i \left(\sum_{v=1}^{\infty} b_{n,v}(x)\{v-1-(n+2)x\}^{2j}\right)^{1/2} \left(\sum_{v=1}^{\infty} b_{n,v}(x)\int_{|t-x|\ge\delta} s_{n,v}(t)\, t^{2\gamma}dt\right)^{1/2}$$

$$= \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i O\left(n^{j/2}\right) O\left(n^{-s/2}\right) \qquad \text{for any } s>0$$

Choosing $s > r$ we get $R_6 = o(1)$. Thus due to arbitrariness of $\varepsilon > 0$ it follows that $R_3 = o(1)$. Also $R_4 \to 0$ as $n \to \infty$ and hence $R_2 = o(1)$. Collecting the estimates of R_1 and R_2, we get the required result.

We now prove in the following theorem the asymptotic formula in simultaneous approximation.

Theorem 3.2 Let $f \in C_\gamma[0,\infty), \gamma > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n[B_n^{(r)}(f,x) - f^{(r)}(x)] = r^2 f^{(r)}(x) + [x(1+r)+2r] f^{(r+1)}(x) + x(1+x) f^{(r+2)}(x).$$

Proof. Using Taylor's expansion of f, we have

$$f(t) = \sum \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2}$$

where $\varepsilon(t,x) \to 0$ as $t \to x$ and $\varepsilon(t,x) = O((t-x)^{\beta}), t \to \infty$ for some $\beta > 0$. Applying Lemma 2.2, we have

$$n[B_n^{(r)}(f(t),x) - f^{(r)}(x)] = n\left[\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)\right]$$

$$+ \left[n\int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^{r+2}dt\right]$$

$$= E_1 + E_2 \qquad \text{(say)}.$$

Now

$$E_1 = n\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^\infty W_n^{(r)}(t,x) t^j dt - n f^{(r)}(x)$$

$$= \frac{f^{(r)}(x)}{r!} n[B_n^{(r)}(t^r,x) - (r!)] + \frac{f^{(r+1)}(x)}{(r+1)!} n\left[(r+1)(-x)B_n^{(r)}(t^r,x) + B_n^{(r)}(t^{r+1},x)\right]$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}n\left[\frac{(r+2)(r+1)}{2}x^2B_n^{(r)}(t^r,x)+(r+2)(-x)B_n^{(r)}(t^{r+1},x)+B_n^{(r)}(t^{r+2},x)\right]$$

It is easily verified from Lemma 2.2 that for each $x\in(0,\infty)$

$$B_n(t^i,x)=\frac{(n+i-1)!(n-i-1)!}{((n-1)!)^2}x^i+i(i-1)\frac{(n+i-2)!(n-i-1)!}{((n-1)!)^2}x^{i-1}+O\left(n^{-2}\right),$$

therefore

$$E_1=nf^{(r)}(x)\left[\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}-1\right]+n\frac{f^{(r+1)}(x)}{(r+1)!}\left[((r+1)(-x)(r!)\left\{\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\right\}\right]$$

$$+\left\{\frac{(n+r)!(n-r-2)!}{((n-1)!)^2}(r+1)!x+r(r+1)\frac{(n+r-1)!(n-r-2)!}{((n-1)!)^2}\right\}$$

$$+n\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)x^2}{2}(r!)\frac{(n+r-1)!(n-r-1)!}{((n-1)!)^2}\right]$$

$$+(r+2)(-x)\left\{\frac{(n+r)!(n-r-2)!}{((n-1)!)^2}(r+1)!x+r(r+1)\frac{(n+r-1)!(n-r-2)!}{((n-1)!)^2}(r!)\right\}$$

$$+\left\{\frac{(n+r+1)!(n-r-3)!}{((n-1)!)^2}\frac{(r+2)!}{2}x^2+(r+1)(r+2)\frac{(n+r)!(n-r-3)!}{((n-1)!)^2}(r+1)!x\right\}+O\left(n^{-2}\right)$$

In order to complete the proof of the theorem it is sufficient to show that $E_2\to 0$ as $n\to\infty$, which can be easily be proved along the lines of the proof of Theorem 3.1 and by using Lemma 2.1, Lemma 2.2 and Lemma 2.5.

This completes the proof of the theorem.

Theorem 3.3 Let $f\in C_\gamma[0,\infty),\gamma>0$ and $0<a<a_1<b_1<b<\infty$. Then for all sufficiently large value of n we have

$$\left\|B_n^{(r)}(f,.)-f^{(r)}\right\|_{C[a_1,b_1]}\le Max\left\{M_3\omega_2(f^{(r)},n^{-1/2},a,b),M_4n^{-1}\|f\|_\gamma\right\}$$

where $< M_3=M_3(r),M_4=M_4(r,f).$

The proof of above theorem easily follows along the lines similar to [4].

REFERENCES

[1] H. Berens and G. G. Lorentz, *Inverse theorem for Bernstein polynomials*, Indiana Univ. Math. J. 21 (1972), 693-708.

[2] S. G. Gal, V. Gupta, *Approximation by complex Szasz-Durrmeyer operators in compact disks*, Acta Math. Sci. 34 B (4) (2014), 1157-1165.

[3] V. Gupta, *Direct estimates for a new general family of Durrmeyer type operators*, Bollettino dell'Unione Matematica Italiana 7 (2015), 279-288.

[4] M.K. Gupta, V. Gupta and M. Kumar, *Direct and inverse estimates for a new family of linear positive operators in simultaneous approximation*, J. Math. Anal. Appl., Vol. 330, (2007), 799-816.

[5] V. Gupta, T. Kim, V. K. Singh and B. Lee, *Rate of simultaneous approximation for Baskakov-Szasz operators*, Proceedings of the Jangjeon Mathematical Society, 14 (2011), No. 3. pp. 267-275.

[6] V. Gupta, N. Deo and X. M. Zeng, *Simultaneous Approximation for Szasz-Mirakian-Stancu-Durrmeyer Operators*, Analysis in Theory and Appl. 29(1) (2013), 1-11.

[7] V. Gupta and P. Gupta, *Direct theorem in simultaneous approximation for Szasz-Mirakyan Baskakov type operators*, Kyungpook Math. J. 41(2) (2001), 243-249.

[8] V. Gupta and R. Yadav, *Direct estimates in simultaneous approximation for BBS operators*, Applied Mathematics and Computation 218 (22) (2012), 11290-11296.

[9] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp. Delhi, 1960, the Russian original appeared in 1959.

[10] C. P. May, *Saturation and inverse theorems for combination of a class of exponential type operators*, Canad. J. Math. 28 (1976), 1224-1250.