



## Deflection and Stress Analysis of Fibrous Composite Laminates

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**Abstract** – First order orthotropic shear deformation equations for the linear elastic bending response of rectangular plates are introduced. Their solution using a computer program in FORTRAN language based on finite differences implementation of the Dynamic Relaxation (DR) method is outlined. The convergence and accuracy of the DR solutions for elastic linear response of isotropic, orthotropic, and laminated plates are established by comparison with various exact and approximate solutions. The present Dynamic Relaxation method (DR) coupled with Finite Differences method (FD) shows a fairly good agreement with other analytical and numerical methods used in the present study. It was found that the DR linear theory program using uniform meshes can be employed in the analysis of different thicknesses and length to side ratios for isotropic, orthotropic and laminated fibrous plates under uniform loads in a fairly good accuracy. These comparisons show that the type of mesh used (i.e. uniform or graded) is responsible for the considerable variations in the mid – side and corner stress resultants. It is found that the convergence of the DR solution depends on several factors including boundary conditions, meshes size, fictitious densities and applied load. It is also found that the DR linear theory can be employed with less accuracy in the analysis of moderately thick and flat isotropic, orthotropic or laminated plates under uniform loads. It is also found that the deflection of the plate becomes of an acceptable value when the length to thickness ratio decreases. For simply supported (SS1) edge conditions, all the comparison results confirmed that deflection depends on the direction of the applied load and the arrangement of the layers.

**Keywords:** Linear analysis, fibrous composites, deflection and stress, shear deformation theory, finite differences

### Notations

$a, b$  plate side lengths in  $x$  and  $y$  directions respectively.

$A_{ij}$  ( $i, j = 1, 2, 6$ ) Plate's in plane stiffnesses.

$A_{44}, A_{55}$  Plate's transverse shear stiffness

$D_{ij}$  ( $i, j = 1, 2, 6$ ) Plate's flexural stiffness

$\varepsilon_x^\circ, \varepsilon_y^\circ, \varepsilon_{xy}^\circ$  Mid – plane direct and shear strains

$\varepsilon_{xz}^\circ, \varepsilon_{yz}^\circ$  Mid – plane transverse shear strains.

$E_1, E_2, G_{12}$  In – plane elastic longitudinal, transverse and shear modulus.

$G_{13}, G_{23}$  Shear modulus in the  $x - z$  and  $y - z$  planes respectively.

$M_x, M_y, M_{xy}$  Stress couples.

$N_x, N_y, N_{xy}$  Stress resultants.

$\bar{q}$  Dimensionless transverse pressure.

$Q_x, Q_y$  Transverse shear resultants.

$u, v$  In – plane displacements.

$w$  Deflections

$\bar{w}$  Dimensionless deflections.

$x, y, z$  Cartesian co – ordinates.

$\delta t$  Time increment.

$\phi, \psi$  Rotations of the normal to the plate mid – plane.

$\nu_{xy}$  Poisson's ratio.

$\ell_u, \ell_v, \ell_w, \ell_\phi, \ell_\psi$  In plane, out of plane and rotational fictitious densities.

$\chi_x^\circ, \chi_y^\circ, \chi_{xy}^\circ$  Curvature and twist components of plate mid – plane.

$\bar{\sigma}_x, \bar{\sigma}_y$  Dimensionless direct stresses in the x and y directions respectively

$\bar{\sigma}_{xy}, \bar{\sigma}_{xz}, \bar{\sigma}_{yz}$  Dimensionless in plane and out of plane shear stresses.

## I. INTRODUCTION

From the point of view of solid mechanics, the deformation of a plate subjected to transverse loading consists of two components: flexural deformation due to rotation of cross-sections, and shear deformation due to sliding of sections or layers. The resulting deformation depends on two parameters: the thickness to length ratio and the ratio of elastic to shear moduli. When the thickness to length ratio is small, the plate is considered thin, and it deforms mainly by flexure or bending; whereas when the thickness to length and the modular ratios are both large, the plate deforms mainly through shear. Due to the high ratio of in-plane modulus to transverse shear modulus, the shear deformation effects are more pronounced in the composite laminates subjected to transverse loads than in the isotropic plates under similar loading conditions.

The three-dimensional theories of laminates in which each layer is treated as homogeneous anisotropic medium as stated by Reddy [1] are intractable as the number of layers becomes moderately large. Thus, a simple two-dimensional theory of plates that accurately describes the global behavior of laminated plates seems to be a compromise between accuracy and ease of analysis. Numerical results obtained using refined finite element analysis in the work of D.J. Vuksanovic [2], and [3] and their comparisons with exact three dimensional analysis pointed out that the higher order theory provides results which are accurate and acceptable for all ranges of thickness and modular ratio.

Putchu and Reddy [4] classified the two-dimensional analyses of laminated composite plates into two categories: (1) the classical lamination theory, and (2) shear deformation theories. In both theories it is assumed that the laminate is in a state of plane stress, the individual lamina is linearly elastic, and there is perfect bonding between layers. The classical laminate theory (CLPT), which is an extension of the classical plate theory (CPT) applied to laminated plates was the first theory formulated for the analysis of laminated plates by Reissner and Stavsky [5] in 1961, in which the Kirchhoff-Love assumption that normal to the mid-surface before deformation remain straight and normal to the mid-surface after deformation is used, but it is not adequate for the flexural analysis of moderately thick laminates. However, it gives reasonably accurate results for many engineering problems i.e. thin composite plates, as stated by Srinivas and Rao [6] and Reissner and Stavsky [5]. This theory ignores the transverse shear stress components and models a laminate as an equivalent single layer. The classical laminated plate theory (CLPT) under-predicts deflections as proved by Turvey and Osman [7], [8], [9] and Reddy [1] due to the neglect of transverse shear strain. The errors in deflections are even higher for plates made of advanced filamentary composite materials like graphite -epoxy and boron-epoxy, whose elastic modulus to shear modulus ratios are very large (i.e. of the order of 25 to 40, instead of 2.6 for typical isotropic materials). However, these composites are susceptible to thickness effects because their effective transverse shear moduli are significantly smaller than the effective elastic modulus along the fiber direction. This effect has been confirmed by Pagano [10] who obtained analytical solutions of laminated plates in bending based on the three-dimensional theory of elasticity. He proved that classical laminated plate theory (CLPT) becomes of less accuracy as the side to thickness ratio decreases. In particular, the deflection of a plate predicted by CLPT is considerably smaller than the analytical value for side to thickness ratio less than 10. These high ratios of elastic modulus to shear modulus render classical laminate theory as inadequate for the analysis of composite plates.

Many theories which account for the transverse shear and normal stresses are available in the literature (see, for example Mindlin [11]). These are too numerous to review here. Only some classical papers and those which constitute a background for the present thesis will be considered. These theories are classified according to Phan and Reddy [12] into two major classes on the basis of the assumed fields as : (1) stress based theories, and (2) displacement based theories. The stress-based theories are derived from stress fields, which are assumed to vary linearly over the thickness of the plate:

$$\sigma_i = \frac{M_i}{(h^2/6)} \cdot \frac{z}{(h/2)} \quad (i=1,2,6) \quad (1)$$

(Where  $M_i$  is the stress couples, h is the plate thickness, and z is the distance of the lamina from the plate mid-plane)

The displacement-based theories are derived from an assumed displacement field as:

$$\begin{aligned} u &= u_o + zu_1 + z^2u_2 + z^3u_3 + \dots \\ v &= v_o + zv_1 + z^2v_2 + z^3v_3 + \dots \\ w &= w_o + zw_1 + z^2w_2 + z^3w_3 + \dots \end{aligned} \quad (2)$$

Where  $u_o, v_o$  and  $w_o$  are the displacements of the middle plane of the plate.

The governing equations are derived using the principle of minimum total potential energy. The theory used in the present work comes under the class of displacement-based theories. Extensions of these theories which include the linear terms in z in u and v and only the constant term in w, to account for higher -order variations and to laminated plates, can be found in the work of Yang, Norris and Stavsky [13], Whitney and Pagano [14], and Phan and Reddy [12]. In this theory which is called first-order shear deformation theory (FSDT), the transverse planes, which are originally normal and straight to the mid-plane of the plate, are assumed to remain straight but not necessarily normal after deformation,

and consequently shear correction factor are employed in this theory to adjust the transverse shear stress, which is constant through thickness. Recently Reddy [15], and Phan and Reddy [12] presented refined plate theories that use the idea of expanding displacements in the powers of thickness co-ordinate. The main novelty of these works is to expand the in-plane displacements as cubic functions of the thickness co-ordinate, treat the transverse deflection as a function of the x and y co-ordinates, and eliminate the functions  $u_2, u_3, v_2$  and  $v_3$  from equation (2) by requiring that the transverse shear stresses be zero on the bounding planes of the plate. Numerous studies involving the application of the first-order theory to bending analysis can be found in the works of Reddy [16], and Reddy and Chao [17].

In order to include the curvature of the normal after deformation, a number of theories known as Higher-order Shear Deformation Theories (HSDT) have been devised in which the displacements are assumed quadratic or cubic through the thickness of the plate. In this aspect, a variationally consistent higher-order theory which not only accounts for the transverse shear deformation but also satisfies the zero transverse shear stress conditions on the top and bottom faces of the plate and does not require shear correction factors was suggested by Reddy [1]. Reddy's modifications consist of a more systematic derivation of displacement field and variationally consistent derivation of the equilibrium equations. The refined laminate plate theory predicts a parabolic distribution of the transverse shear stresses through the thickness, and requires no shear correction coefficients.

In the non-linear analysis of plates considering higher-order theory (HSDT), shear deformation has received considerably less attention compared with linear analysis. This is due to the geometric non-linearity which arises from finite deformations of an elastic body and which causes more complications in the analysis of composite plates. Therefore fiber-reinforced material properties and lamination geometry have to be taken into account. In the case of anti-symmetric and unsymmetrical laminates, the existence of coupling between bending and stretching complicates the problem further.

Non-linear solutions of laminated plates using higher-order theories have been obtained through several techniques, i.e. perturbation method as in [18], finite element method as in [4], the increment of lateral displacement method as in [19], and the small parameter method as in [20].

In the present work, a numerical method known as Dynamic Relaxation (DR) coupled with finite differences is used. The DR method was first proposed in 1960s; see Rushton [21], Cassell and Hobbs [22], Day [23]. In this method, the equations of equilibrium are converted to dynamic equation by adding damping and inertia terms. These are then expressed in finite difference form and the solution is obtained through iterations. The optimum damping coefficient and time increment used to stabilize the solution depend on a number of factors including the stiffness matrix of the structure, the applied load, the boundary conditions and the size of the mesh used, etc...

## II. SMALL DEFLECTION THEORIES

The equilibrium, strain, constitutive equations and boundary conditions are introduced below without derivation.

### 2.1. Equilibrium equations

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= 0 \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0 \end{aligned} \quad (1)$$

### 2.2. Strain equations

The small deflection strains of the mid – plane of the plate are as given below:

$$\begin{aligned} \epsilon_x^\circ &= \frac{\partial u^\circ}{\partial x} + z \frac{\partial \phi}{\partial x} \\ \epsilon_y^\circ &= \frac{\partial v^\circ}{\partial y} + z \frac{\partial \psi}{\partial y} \\ \epsilon_{xy}^\circ &= \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} + z \left\{ \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right\} \end{aligned} \quad (2)$$

$$\varepsilon_{xz}^{\circ} = \frac{\partial w}{\partial y} + \psi$$

$$\varepsilon_{yz}^{\circ} = \frac{\partial w}{\partial x} + \phi$$

### 2.3. The constitutive equations

The laminate constitutive equations can be represented in the following form:

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{Bmatrix} \varepsilon_j^{\circ} \\ \chi_j^{\circ} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz}^{\circ} \\ \varepsilon_{xz}^{\circ} \end{Bmatrix}$$
(3)

Where  $N_i$  denotes  $N_x, N_y, N_{xy}$  and  $M_i$  denotes  $M_x, M_y, M_{xy}$ .  $A_{ij}, B_{ij}$  and  $D_{ij}$ , ( $i, j = 1, 2, 6$ ) are respectively the membrane rigidities, coupling rigidities and flexural rigidities of the plate.

$\chi_j^{\circ}$  Denotes  $\frac{\partial \phi}{\partial x}, \frac{\partial \psi}{\partial y}$  and  $\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$ .  $A_{44}, A_{45}$  and  $A_{55}$  denote the stiffness coefficients, and are calculated as follows:

$$A_{ij} = \sum_{k=1}^n K_i K_j \int_{\bar{z}_k}^{\bar{z}_{k+1}} c_{ij} dz, (i, j = 4, 5)$$

Where  $c_{ij}$  the stiffness of a lamina and is referred to the plate principal axes and  $K_i, K_j$  are the shear correction factors.

### 2.4. Boundary conditions

All of the analyses described in this paper have been undertaken assuming the plates to be subjected to identical support conditions in the flexural and extensional senses along all edges. The three sets of edge conditions used here are designated as SS1, SS2 and SS3 and are shown in Fig. (1) Below:

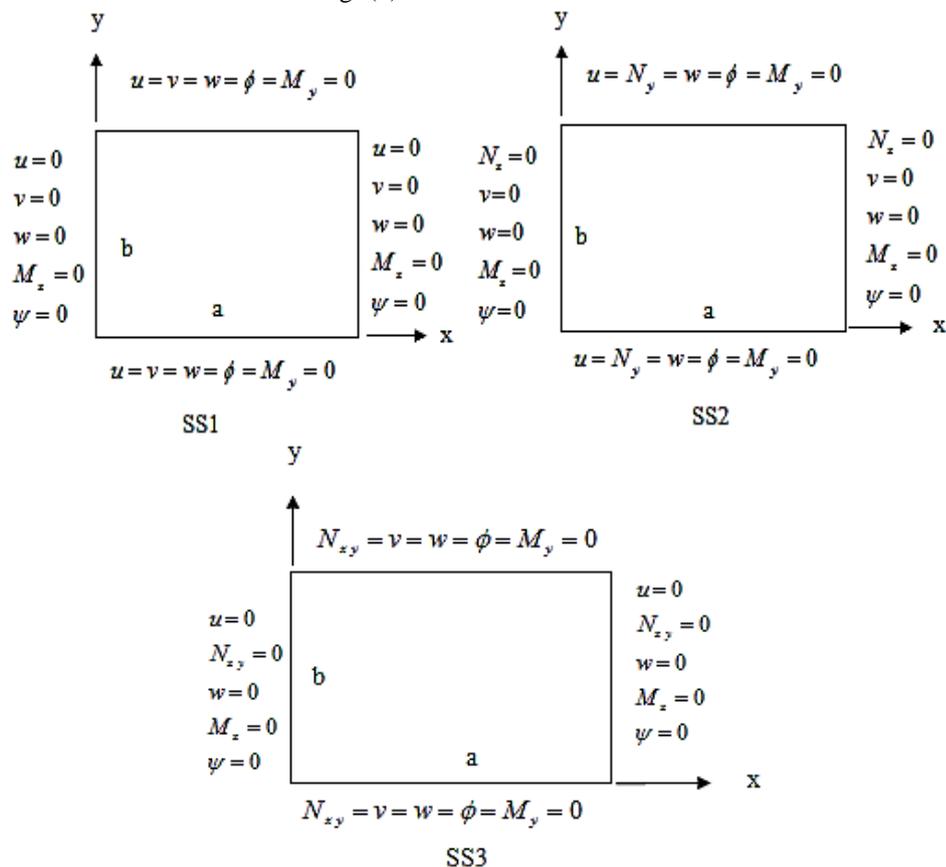


Fig. (1) Simply supported boundary conditions

### III. DYNAMIC RELAXATION OF THE PLATE EQUATIONS

In the present work, a numerical method known as Dynamic Relaxation (DR) coupled with finite differences is used. The DR method was first proposed in 1960s and then passed through a series of studies to verify its validity. In this method, the equations of equilibrium are converted to dynamic equations by adding damping and inertia terms. These are then expressed in finite difference form and the solution is obtained through iterations.

Numerical techniques other than the DR include finite element method, which is widely used in several studies i.e. of Damodar R. Ambur et al [24], Ying Qing Huang et al [25], Onsy L. Roufaeil et al [26]... etc. In a comparison between the DR and the finite element method, Aalami [27] found that the computer time required for finite element method is eight times greater than for DR analysis, whereas the storage capacity for finite element analysis is ten times or more than that for DR analysis. This fact is supported by Putcha and Reddy [4] who they noted that some of the finite element formulation requires large storage capacity and computer time. The errors inherent in the DR technique ([28] – [39]) include discretization error which is due to the replacement of a continuous function with a discrete function, and there is an additional error because the discrete equations are not solved exactly due to the variations of the velocities from the edge of the plate to the center. Finer meshes reduce the discretization error, but increase the round – off error due to the large number of calculations involved. Hence due to less computations and computer time involved in the present study, the DR method is considered more suitable than the finite element method.

The plate equations are written in dimensionless forms. Damping and inertia terms are added to Eqns. (1). Then the following approximations are introduced for the velocity and acceleration terms:

$$\frac{\partial \alpha}{\partial t} = \frac{1}{2} \left[ \frac{\partial \alpha^a}{\partial t} + \frac{\partial \alpha^b}{\partial t} \right]$$

$$\frac{\partial^2 \alpha}{\partial t^2} = \left( \frac{\partial \alpha^a}{\partial t} - \frac{\partial \alpha^b}{\partial t} \right) / \delta t$$
(4)

In which  $\alpha \equiv u, v, w, \phi, \psi$ . Hence Eqns(1) becomes :

$$\frac{\partial u^a}{\partial t} = \frac{1}{1+k_u^*} \left[ (1-k_u^*) \frac{\partial u^b}{\partial t} + \frac{\delta t}{\ell_u} \left( \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \right]$$

$$\frac{\partial v^a}{\partial t} = \frac{1}{1+k_v^*} \left[ (1-k_v^*) \frac{\partial v^b}{\partial t} + \frac{\delta t}{\ell_v} \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \right]$$

$$\frac{\partial w^a}{\partial t} = \frac{1}{1+k_w^*} \left[ (1-k_w^*) \frac{\partial w^b}{\partial t} + \frac{\delta t}{\ell_w} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q \right) \right]$$

$$\frac{\partial \phi^a}{\partial t} = \frac{1}{1+k_\phi^*} \left[ (1-k_\phi^*) \frac{\partial \phi^b}{\partial t} + \frac{\delta t}{\ell_\phi} \left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \right) \right]$$

$$\frac{\partial \psi^a}{\partial t} = \frac{1}{1+k_\psi^*} \left[ (1-k_\psi^*) \frac{\partial \psi^b}{\partial t} + \frac{\delta t}{\ell_\psi} \left( \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y \right) \right]$$
(5)

In Eqns. (4) and (5) the superscripts a and b refer respectively to the values of velocities after and before the time step  $\delta t$ , and  $k_\alpha^* = \frac{1}{2} k_\alpha \delta t \rho_\alpha^{-1}$ . The displacements at the end of each time increment,  $\delta t$ , are evaluated using the following simple integration procedure:

$$\alpha^a = \alpha^b + \delta t \frac{\partial \alpha^b}{\partial t}$$
(6)

The complete equation system is represented by eqns. (5), (6), (2) and (3). The DR algorithm operates on these equations is as follows:

- Step 1: set initial conditions (usually all variables are zero).
- Step 2: compute velocities from eqns. (5).
- Step 3: compute displacements from eqn. (6).
- Step 4: Apply displacement boundary conditions.
- Step 5: compute strains from eqns. (2).
- Step 6: compute stress resultants, etc. from eqns. (3).
- Step 7: Apply stress resultants ... etc. boundary conditions.
- Step 8: check if velocities are acceptably small (say  $< 10^{-6}$ ).
- Step 9: If Step 8 is satisfied print out results, otherwise repeat steps from (2 – 8).

**IV. RESULTS AND DISCUSSIONS**

New numerical results of the DR program were obtained. The present DR results are compared with similar results either generated by other DR techniques or another alternative techniques including approximate analytical and exact solutions. In the following discussion a variety of linear deflections and stresses are dealt with including isotropic, orthotropic, and laminated plates subjected to static uniformly distributed load. The analysis uses first order shear deformation theory to study the behavior of the linear theory of the plate equations.

New numerical results were obtained to select a suitable mesh size of the plate. Table (1) shows variations of the central deflections through different sizes of mesh for an isotropic, moderately thick plate ( $h/a = 0.1$ ), simply supported (SS1). These results suggest that a 5 x 5 mesh size over one quarter of the plate is quite enough for the present work (i.e. less than 0.3% error compared to the finest mesh available). In table (2) new results of the DR program were generated. The comparison of the present DR deflections and stresses with that generated by Turvey and Osman [7] and Reddy [15] is presented for a uniformly loaded isotropic plate of thin (i.e.  $h/a=0.01$ ), moderately thick (i.e.  $h/a=0.1$ ), and thick laminates (i.e.  $h/a=0.2$ ) using simply supported condition (SS1). The present DR results of central deflections and stresses showed good agreement with the other results even though the plate is square or rectangle. Another comparison analysis for small deformations of thin and moderately thick simply supported square isotropic plates (SS1) between the present DR method, and Roufaeil [26] of two and three nodes strip method is shown in table (3). Again, these results provide further confirmation that the present DR analysis based on a 5 x 5 quarter – plate mesh produces results of acceptable accuracy.

In the following analyses, several orthotropic materials were employed; their properties are given in table (4). Exact FSDT solutions are available for plates simply supported on all four edges (SS2). By imposing only a small load on the plate, the DR program may be made to simulate these small deflections. In table (5), the computations were made for uniform loads and for thickness / side ratios ranging from 0.2 to 0.01 of square simply supported in – plane free plates made of material I with ( $\bar{q} = 1.0$ ). In this case the central deflections of the present DR method are close to those of Turvey and Osman [8], and Reddy [15]. Another small deflection analysis is shown on table (6), and it was made for uniformly loaded plates with simply supported in – plane fixed condition (SS1) of material II and subjected to uniform loading ( $\bar{q} = 1.0$ ). In this analysis, the four sets of results are the same for the central deflections and stresses at the upper and lower surfaces of the plate and also the same for the mid – plane stresses. Nevertheless, the exact solution of Srinivas and Rao [6] is not in a good agreement with the others as far as stresses are concerned. These differences may be attributed to the exact solution theory adopted in [6].

Most of the published literature on laminated plates are devoted to linear analysis and in particular to the development of higher order shear deformation theories. Comparatively, there are few studies on the nonlinear behavior of laminated plates and even fewer are those which include shear deformations. The elastic properties of the material used in the analyses are given in table (4). The shear correction factors are  $k_4^2 = k_5^2 = 5/6$ , unless otherwise stated.

In table (7) which show a comparison between the present DR method and finite element results [16] for a simply supported condition (SS3) plate. There are four antisymmetric angle ply laminates of material III which are subjected to a small uniform load ( $\bar{q} = 1.0$ ). The central deflections and stresses are recorded for different thickness ratios including thick, moderately thick, and thin laminates. These results are compared with Reddy’s finite element results [16] and are found in a good agreement despite the different theory adopted in the latter case.

Another comparison analysis of central deflections between the present DR method, Zenkour et al [20] using third order shear deformation theory and Librescu et al [40] which are made of material IV are illustrated in table (8). The three results showed a good agreement especially as the thickness to length ratio decreases.

Table 1 DR solution convergence results for a simply supported (SS1) square plate subjected to uniform pressure ( $\bar{q} = 1.0, \frac{h}{a} = 0.1$  and  $\nu = 0.3$ ).

Mesh size	$\bar{w}_c$
2x2	0.04437
3x3	0.04592
4x4	0.04601
5x5	0.04627
6x6	0.04629
7x7	0.04638
8x8	0.04640

Table 2 Comparisons of present DR, Turvey and Osman [7], and exact values of Reddy [15] small deflection results for uniformly loaded simply supported (SS1) square and rectangular plates of various thickness ratios  $\bar{q} = 1.0, \nu = 0.3$

$a/b$	$h/a$	$S$	$\bar{w}_c$	$\bar{\sigma}_x(1)$	$\bar{\sigma}_y(1)$	$\bar{\sigma}_{xy}(2)$	$\bar{\sigma}_{xz}(3)$	$\bar{\sigma}_{yz}(4)$
1	0.20	1	0.0529	0.2879	0.2879	- 0.2035	0.3983	0.3983

		2	0.0529	0.2879	0.2879	- 0.2035	0.3984	0.3984	
		3	0.0536	0.2873	0.2873	- 0.1946	0.3928	0.3928	
		0.10	1	0.0463	0.2866	0.2866	- 0.2038	0.3983	0.3960
		0.10	2	0.0463	0.2865	0.2865	- 0.2038	0.3990	0.3990
			3	0.0467	0.2873	0.2873	- 0.1946	0.3928	0.3928
			0.01	1	0.0440	0.2853	0.2853	- 0.2033	0.3960
		0.01	2	0.0441	0.2860	0.2860	- 0.2039	0.3990	0.3990
			3	0.0444	0.2873	0.2873	- 0.1946	0.3928	0.3928
			2	0.20	1	0.1204	0.2825	0.6165	- 0.2952
2	0.1216	0.2840			0.6225	- 0.2829	0.4341	0.5410	
3	0.1248	0.2779			0.6100	- 0.2769	0.4192	0.5451	
0.10	1	0.1111		0.2819	0.6148	- 0.2964	0.4200	0.5412	
	2	0.1122		0.2838	0.6209	- 0.2843	0.4358	0.5447	
	3	0.1142		0.2779	0.6100	- 0.2769	0.4192	0.5451	
0.01	1	0.1080		0.2823	0.6141	- 0.2970	0.4200	0.5400	
	2	0.1109		0.2842	0.6212	- 0.2857	0.4377	0.5472	
	3	0.1106		0.2779	0.6100	- 0.2769	0.4192	0.5451	

S (1): present DR results

S (2): DR results of [7]

S (3): Exact results of [15]

$$(1)x = \frac{1}{2}a, y = \frac{1}{2}b, z = \frac{1}{2}h; (2)x = y = 0, z = \frac{1}{2}h; (3)x = 0, y = \frac{1}{2}b, z = 0; (4)x = \frac{1}{2}a, y = z = 0$$

Table 3 Dimensionless central deflection of a square simply supported isotropic plate (SS1) ( $\bar{q} = 1.0, \nu = 0.3, k^2 = 0.833$ )

$a/h$	Present DR Results	3 – node strip [26]	2 – node strip [26]
100	0.00403	0.00406	0.00406
10	0.00424	0.00427	0.00426

Table 4 Material properties used in the orthotropic and laminated plate Comparison analysis

Material	$E_1/E_2$	$G_{12}/E_2$	$G_{13}/E_2$	$G_{23}/E_2$	$\nu_{12}$	$SCF (k_4^2 = k_5^2)$
I	25.0	0.5	0.5	0.2	0.25	5/6
II	1.904	0.558	0.339	0.566	0.44	5/6
III	40.0	0.5	0.5	0.5	0.25	5/6
IV	12.308	0.526	0.526	0.335	0.24	5/6

Table 5 Comparison of present DR, Turvey and Osman [8], and Reddy [15] center deflections of a simply supported (SS2) square orthotropic plate made of material I for different thickness ratios when subjected to uniform loading ( $\bar{q} = 1.0$ ).

Thickness ratio $h/a$	Uniform Loading		
	$\bar{w}_c (DR)$ present	$\bar{w}_c (DR)$ [8]	$\bar{w}_c (exact)$ [15]
0.2	0.017914	0.017912	0.018159
0.1	0.009444	0.009441	0.009519
0.08	0.008393	0.008385	0.008442
0.05	0.007245	0.007230	0.007262
0.02	0.006617	0.006602	0.006620
0.01	0.006512	0.006512	0.006528

Table 6 comparison of present DR, Turvey and Osman [8], Reddy [16], and exact solution [6] for a uniformly loaded simply supported (SS1) orthotropic plate made of material II when subjected to uniform loading ( $\bar{q} = 1.0$ ).

$b/a$	$h/a$	$S$	$\bar{w}_c$	$\bar{\sigma}_x(1)$	$\bar{\sigma}_{xz}(2)$
1	0.05	1	0.0306	0.3563	0.4387
		2	0.0306	0.3562	0.4410
		3	0.0308	0.3598	0.4351
		4	0.0308	0.3608	0.5437
	0.10	1	0.0323	0.3533	0.4393
		2	0.0323	0.3534	0.4395
		3	0.0326	0.3562	0.4338
		4	0.0325	0.3602	0.5341
	0.14	1	0.0344	0.3498	0.4367
		2	0.0344	0.3498	0.4374
		3	0.0347	0.3516	0.5328
		4	0.0346	0.3596	0.5223
2	0.05	1	0.0629	0.6569	0.6506
		2	0.0629	0.6568	0.5637
		3	0.0636	0.6550	0.5600
		4	0.0636	0.6567	0.7024
	0.10	1	0.0657	0.6566	0.5623
		2	0.0657	0.6566	0.5628
		3	0.0665	0.6538	0.5599
		4	0.0664	0.6598	0.6927
	0.14	1	0.0692	0.6564	0.5613
		2	0.0692	0.6564	0.5613
		3	0.0703	0.6521	0.5597
		4	0.0701	0.6637	0.6829

S (1): present DR results

S (2): DR results of [8]

S (3): Finite element solution [16]

S (4): Exact solution [6]

$$(1) x = \frac{1}{2}a, y = \frac{1}{2}b, z = \frac{1}{2}h; (2) x = 0, y = \frac{1}{2}b, z = 0$$

Table 7 Comparison of present DR and Reddy finite element results [16] for  $[45^\circ/-45^\circ/45^\circ/-45^\circ]$  simply supported (SS3) square laminate made of material III and subjected to uniform loads and for different thickness ratios ( $\bar{q} = 1.0$ ).

$h/a$	$S$	$\bar{w}_c \times 10^3$	$\bar{\sigma}_x(1)$
0.20	1	9.0809	0.2022
	2	9.0000	0.1951
0.10	1	4.3769	0.2062
	2	4.2000	0.1949
0.05	1	3.2007	0.2081
	2	3.0000	0.1938
0.04	1	3.0574	0.2090
	2	2.9000	0.1933
0.02	1	2.8371	0.2063
	2	2.8000	0.1912

S (1): present DR results.

S (2): Reddy [16] as read from graph. (1)  $x = y = \frac{1}{2}a, z = \frac{1}{2}h$

Table 8 Non – dimensionalized deflections in three layers cross – ply  $[0^\circ/90^\circ/45^\circ/0^\circ]$  simply supported (SS1) square laminates of material IV under uniform load ( $\bar{q} = 1.0$ ).

$a/h$	$S$	$\bar{w}_c$
2	1	0.0693
	2	0.0726
	3	0.0716

5	1	0.0224
	2	0.0232
	3	0.0235
10	1	0.0147
	2	0.0150
	3	0.0151
20	1	0.0127
	2	0.0128
	3	0.0128

S (1): present DR results linear analysis

S (2): Librescu L and Khdeir A.A [40]

S (3): A.M. Zenkour, and M.E. Fares [20] results.

## V. CONCLUSIONS

A Dynamic relaxation (DR) program based on finite differences has been developed for small deflection analysis of rectangular laminated plates using first order shear deformation theory (FSDT). The displacements are assumed linear through the thickness of the plate. A series of new results for uniformly loaded thin, moderately thick, and thick plates with simply supported edges have been presented and compared with other exact and approximate solutions so as to validate the accuracy of the present DR program. These comparisons show that the type of mesh used (i.e. uniform or graded) is responsible for the considerable variations in the mid – side and corner stress resultants. It is found that the convergence of the DR solution depends on several factors including boundary conditions, meshes size, fictitious densities and applied load. It is also found that the DR linear theory can be employed with less accuracy in the analysis of moderately thick and flat isotropic, orthotropic or laminated plates under uniform loads. It is also found that the deflection of the plate becomes of an acceptable value when the length to thickness ratio decreases. For simply supported (SS1) edge conditions, all the comparison results confirmed that deflection depends on the direction of the applied load and the arrangement of the layers.

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