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Common Fixed Point Theorems in Menger Space using Semi-Compatibility

Dr. Varsha Sharma
Assistant Professor, Department of Mathematics, Institute of Engineering & Science, IPS Academy, Rajendra Nagar, Indore, Madhya Pradesh, India

Abstract: ***The aim of this paper is to prove a common fixed point theorem for six mappings on Menger space using notion of semi-compatibility and reciprocal continuity of maps satisfying an implicit relation . Our result generalizes and extends many known results in menger spaces and metric spaces.***

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I. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [2]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [6] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [7] obtained Banach contraction principal in a complete Menger space, which is a milestone in developing fixed point theory in Menger space. Sessa [8] initiated the tradition of improving coomutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [1] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [3]. Pant [4] introduced the notion of reciprocal continuity of mappings in metric spaces. Popa [5] proved theorem for weakly compatible non-continuous mapping using implicit relation. Recently Singh and Jain [9] have been introduced semi-compatible , compatible and weak compatible maps in Menger space.

The purpose of this paper is to prove a common fixed point theorem in Menger space using weak compatibility, semi-compatibility , an implicit relation and reciprocal continuity. Here we generalize the result of [10] by

(1) Increasing the number of self maps from four to six .
(2) Using the notion of reciprocal continuity .

II. PRELIMINARIES

Definition 2.1. A mapping $F : R \rightarrow R^+$ is called a distribution if it is non-decreasing left continous with inf $\{F(t) : t \in R\} = 0$ and sup $\{F(t) : t \in R\} = 1$. We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defiend by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Definition 2.2. A Probabilistic metric space (PM- space) is an ordered pair (X , F), where X is an abstract set of elements and $F : X \times X \rightarrow L$, defined by (p,q) $\rightarrow F_{p,q}$, where L is the set of all distribution functions i.e. L = { $F_{p,q}$ / p,q $\in$ X}, if the functions Fp,q satisfy:

(a) $F_{p,q}(x) = 1$, for all x >0, if and only if p = q;
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q} = F_{q,p}$;
(d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 2.3. A mapping $t : [0,1] \times [0,1] \rightarrow [0,1]$ is called a t-norm if

(a) $t(a,1) = a$;
(b) $t(a,b) = t(b,a)$;
(c) $t(c,d) \geq t(a,b)$ for $c \geq a$, $d \geq b$;
(d) $t(t(a,b),c) = t(a,t(b,c))$,
for all a,b,c,d $\in$ [0,1].

Definition 2.4. A Menger space is a triplet (X,F,t) where (X,F) is PM-space and t is a t-norm such that $\forall$ p,q,r $\in$ X and $\forall$ x,y ≥ 0

$$F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y)).$$

Schweizer and Sklar [6] proved that if (X, F,t) is a Menger space with $\sup_{0<x<1} t(x,x) = 1$, then (X,F,t) is a Hausdorff topological space in the topology induced by the family of (ε,λ)-neighborhoods, $\{U_p(\varepsilon,\lambda) : p \in X, \varepsilon > 0, \lambda > 0\}$, where $U_p(\varepsilon,\lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1 - \lambda\}$.

Definition 2.5. Let (X,F,t) be a Menger space with $\sup_{0<x<1} t(x,x) = 1$. A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \to p$) if for every $\varepsilon > 0$ and $\lambda > 0$, $\exists$ an integer $M(\varepsilon,\lambda)$ such that $F_{pn,p}(\varepsilon) > 1 - \lambda$, $\forall$ $n \geq M(\varepsilon,\lambda)$. Further , the sequence is said to be a cauchy sequence if for each $\varepsilon > 0$ and $\lambda > 0$, $\exists$ an integer $M(\varepsilon,\lambda)$ such that $F_{pn,pm}(\varepsilon) > 1 - \lambda$, $\forall$ $n,m \geq M(\varepsilon,\lambda)$. A Menger space (X,F,t) is said to be complete if every cauchy sequence in it converges to a point of it.
A complete metric space can be treated as a complete Menger space in the following way.

Proposition 2.6. If (X , d) is a metric space then the metric d induces a mapping $X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p,q))$, $\forall p,q \in X$ and $x \in R$. Further, if $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = \min\{a,b\}$, then (X,F,t) is a Menger space. It is complete if (X,d)is complete. Then space (X, F, t) so obtained is called the induced Menger space.

Proposition 2.7. In a Menger space (X,F,t),if $t(x,x) \geq x, \forall x \in [0,1]$ then $t(a,b) = \min\{a,b\}$, $\forall a,b \in [0,1]$.

Definition 2.8. Self mappings A and S of a Menger space(X,F,t) are said to be weak compatible if they commute at their coincidence points i.e. Ax = Sx for $x \in X$ implies ASx = SAx.

Definition 2.9. Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASpn,SApn}(x) \to 1$, $\forall x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.
Here we introduce the notion of semi-compatible mappings in Menger space.

Definition 2.10. Self – mappings A and S of a Manger space (X,F,t) are called semi-compatible if $F_{ASpn,Su}(x) \to 1$, $\forall x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.

Proposition 2.11. If selfmappings A and S of a Menger space (X,F,t) are semi-compatible then they are weak compatible.

Proposition 2.12. Let S and T be two self maps on a Menger space (X,F,t) with $t(a,a) \geq a$, $\forall a \in [0,1]$ of which T is continuous. Then (S,T) is semi-compatible if and only if (S,T) is compatible.
In [9] it has been shown that the semi-compatibility of (A,S) need not imply the semi-compatibility of (S,A). Further , an example of pair of self maps is given , which is commuting (hence compatible, weak compatible) yet it is not semi-compatible.

Lemma 2.13. Let $\{p_n\}$ be a sequence in a Menger space (X,F,t) with continuous t-norm $t(x,x) \geq x$, $\forall x \in [0,1]$. If $\exists$ $k \in (0,1)$ such that for all $x > 0$ and $n \in N$, $F_{pn,pn+1}(kx) \geq F_{pn-1,pn}(x)$. Then $\{p_n\}$ is a Cauchy sequence in X.

Definition 2.14. Let A and S be mappings from a Menger space (X,F,t) into itself . Then the mappings are said to be reciprocally continuous if

$$\lim_{n\to\infty} ASx_n = Ax \quad \text{and} \quad \lim_{n\to\infty} SAx_n = Sx$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$$

If A and S are both continuous then they are obviously reciprocally continuous but the converse is not true .

A Class of Implicit Relation: Let Φ be set of all real continuous functions $\phi : (R^+)^4 \to R$, nondecreasing in first argument and satisfying the following conditions:

(i) For $u,v \geq 0$, $\phi(u,v,v,u) \geq 0$ or $\phi(u,v,u,v) \geq 0$ imply $u \geq v$.
(ii) $\phi(u,u,1,1) \geq 0$ implies $u \geq 1$.

Example 2.14. Define $\phi(t_1,t_2,t_3,t_4) = 15t_1-13t_2+5t_3-7t_4$. Then $\phi \in \Phi$.

III. MAIN RESULT

Theorem 3.1 . Let A, B, S, T, I and J be self-mappings of a complete Menger space (X , F , Min) such that

(a) $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$
(b) the pair (AB , I) is semi-compatible and (ST , J) is weak compatible
(c) the pair (AB , I) is reciprocally continuous .

For some $\phi \in \Phi$, there exists $k \in (0 , 1)$ such that for all $x , y \in X$ and $t > 0$,

$$\phi (F_{ABx , STy}(kt) , F_{Ix , Jy}(t) , F_{ABx , Ix}(t) , F_{STy , Jy}(kt)) \geq 0 \quad (1)$$

$$\phi (F_{ABx , STy}(kt) , F_{Ix , Jy}(t) , F_{ABx , Ix}(kt) , F_{STy , Jy}(t)) \geq 0 \quad (2)$$

Then AB , ST , I and J have a unique common fixed point . Furthermore , if the pairs (A , B), (A , I) , (B , I) , (S , T) , (S , J) and (T , J) are commuting mapping then A , B , S , T , I and J have a unique common fixed point .

Proof . Let x_0 be an arbitrary point in X. Since $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$, there exist x_1 , $x_2 \in X$ such that $ABx_0 = Jx_1$, $STx_1 = I x_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = ABx_{2n} = Jx_{2n+1} , \quad y_{2n+2} = STx_{2n+1} = Ix_{2n+2}$$

for n = 0,1,2,…. . Now making in (1) $x = x_{2n}$, $y = x_{2n+1}$, we obtain

$$\phi (F_{ABx_{2n} ,STx_{2n+1}}(kt) , F_{Ix_{2n} ,Jx_{2n+1}}(t) , F_{ABx_{2n} ,Ix_{2n}}(t) , F_{STx_{2n+1} , Jx_{2n+1}}(kt)) \geq 0$$

that is

$$\phi (F_{y_{2n+1},y_{2n+2}}(kt) , F_{y_{2n} ,y_{2n+1}}(t) , F_{y_{2n+1},y_{2n}}(t) , F_{y_{2n+2},y_{2n+1}}(kt)) \geq 0$$

Using (i) we get

$$F_{y_{2n+2} ,y_{2n+1}}(kt) \geq F_{y_{2n+1} , y_{2n}}(t) \quad (3)$$

Analogous , putting $x = x_{2n+2}$, $y = x_{2n+1}$ in (2) , we have

$$\phi (F_{y_{2n+3},y_{2n+2}}(kt) , F_{y_{2n+1} ,y_{2n+2}}(t) , F_{y_{2n+3},y_{2n+2}}(kt) , F_{y_{2n+1},y_{2n+2}}(t)) \geq 0$$

Using (i) , we get

$$F_{y_{2n+3},y_{2n+2}}(kt) \geq F_{y_{2n+1} ,y_{2n+2}}(t) \quad (4)$$

Thus , from (3) and (4) , for any n and t , we have

$$F_{y_n ,y_{n+1}}(kt) \geq F_{y_{n-1} ,y_n}(t) . \quad (5)$$

Hence by lemma 2.13 , $\{y_n\}$ is a Cauchy sequence in X , which is complete .Therefore $\{y_n\}$ converges to $p \in X$. The sequences $\{ ABx_{2n}\}$, $\{ STx_{2n+1}\}$, $\{ Ix_{2n}\}$, $\{ Jx_{2n+1}\}$, being subsequences of $\{y_n\}$ also converges to p , that is

$$\{ ABx_{2n}\} \rightarrow p , \quad \{ STx_{2n+1}\} \rightarrow p \quad (6)$$

$$\{ Ix_{2n}\} \rightarrow p , \quad \{ Jx_{2n+1}\} \rightarrow p \quad (7)$$

The reciprocal continuity of the pair (AB , I) gives

$$ABIx_{2n} \rightarrow ABp \quad \text{and} \quad IABx_{2n} \rightarrow Ip$$

The semi-compatibility of the pair (AB , I) gives

$$\lim_{n\rightarrow\infty} ABIx_{2n} = Ip$$

From the uniqueness of the limit in a fuzzy metric space, we obtain that $ABp = Ip$ (8)

Step1. By putting $x = p$, $y = x_{2n+1}$ in (1) , we obtain

$$\phi (F_{ABp, STx_{2n+1}}(kt) , F_{Ip, Jx_{2n+1}}(t) , F_{ABp, Ip}(t) , F_{STx_{2n+1} , Jx_{2n+1}}(kt)) \geq 0$$

Letting n tends to infinity and using (6) , (7) and (8) we get

$$\phi (F_{Ip, p}(kt) , F_{Ip, p}(t) , F_{Ip, Ip}(t) , F_{p, p}(kt)) \geq 0$$

As ϕ is non decreasing in first argument , we have

$$\phi (F_{Ip, p}(t) , F_{Ip, p}(t) , 1 , 1) \geq 0$$

Using (ii) , we have $F_{Ip, p}(t) \geq 1$ for all $t > 0$, which gives $F_{Ip, p}(t) = 1$, that is

$$I\,p = p = AB\,p \qquad (9)$$

Step 2. As $AB(X) \subset J(X)$, there exists $u \in X$ such that $AB\,p = I\,p = p = J\,u$.

Putting $x = x_{2n}$, $y = u$ in (1) we obtain that

$$\phi\,(F_{ABx_{2n},\,STu}(kt)\,,\,F_{Ix_{2n},\,Ju}(t)\,,\,F_{ABx_{2n},\,Ix_{2n}}(t)\,,\,F_{STu,\,Ju}(kt)) \geq 0$$

Letting n tends to infinity and using (6) and (7) we get

$$\phi\,(F_{p,\,STu}(kt)\,,\,1\,,\,1\,,\,F_{STu,\,p}(kt)) \geq 0$$

Using (i), we have $F_{p,\,STu}(kt) \geq 1$ for all $t > 0$, which gives $F_{p,\,STu}(kt) = 1$.
Thus $p = ST\,u$. Therefore, $ST\,u = J\,u = p$. Since (ST, J) is weak compatible, we get
$JST\,u = STJ\,u$, that is,

$$ST\,p = J\,p \qquad (10)$$

Step 3. By putting $x = p$, $y = p$ in (1) and using (9) and (10), we obtain

$$\phi\,(F_{ABp,\,STp}(kt)\,,\,F_{Ip,\,Jp}(t)\,,\,F_{ABp,\,Ip}(t)\,,\,F_{STp,\,Jp}(kt)) \geq 0 \text{ that is,}$$

$$\phi\,(F_{ABp,\,STp}(kt)\,,\,F_{ABp,\,STp}(t)\,,\,1\,,\,1) \geq 0.$$

As ϕ is non decreasing in first argument, we have

$$\phi\,(F_{ABp,\,STp}(t)\,,\,F_{ABp,\,STp}(t)\,,\,1\,,\,1) \geq 0$$

Using (ii), we have $F_{ABp,\,STp}(t) \geq 1$ for all $t > 0$, which gives $F_{ABp,\,STp}(t) = 1$
Thus $AB\,p = ST\,p$.
Therefore, $p = AB\,p = ST\,p = I\,p = J\,p$, that is p is a common fixed point of AB, ST, I and J.
Uniqueness. Let q be another common fixed point of AB, ST, I and J. Then $q = AB\,q = ST\,q = I\,q = J\,q$.
By putting $x = p$ and $y = q$ in (1), we get

$$\phi\,(F_{ABp,\,STq}(kt)\,,\,F_{Ip,\,Jq}(t)\,,\,F_{ABp,\,Ip}(t)\,,\,F_{STq,\,Jq}(kt)) \geq 0$$

that is, $\phi\,(F_{p,q}(kt)\,,\,F_{p,q}(t)\,,\,1\,,\,1) \geq 0$.
As ϕ is non decreasing in first argument, we have

$$\phi\,(F_{p,q}(t)\,,\,F_{p,q}(t)\,,\,1\,,\,1) \geq 0$$

Using (ii), we have $F_{p,q}(t) \geq 1$ for all $t > 0$, which gives $F_{p,q}(t) = 1$, that is $p = q$.
Therefore, p is the unique common fixed point of the self-maps AB, ST, I and J.
Finally, we need to show that p is also a common fixed point of A, B, S, T, I and J. For this let p be the unique common fixed point of both the pairs (AB, I) and (ST, J). Then by using commutativity of the pair (A, B), (A, I) and (B, I), we obtain
$A\,p = A(AB\,p) = A(BA\,p) = AB(A\,p)$, $A\,p = A(I\,p) = I(A\,p)$,
$B\,p = B(AB\,p) = B(A(B\,p)) = BA(B\,p) = AB(B\,p)$, $B\,p = B(I\,p) = I(B\,p)$,
Which shows that A p and B p are common fixed point of (AB, I), yielding thereby

$$A\,p = p = B\,p = I\,p = AB\,p \qquad (11)$$

in the view of uniqueness of the common fixed point of the pair (AB, I). Similarly using the commutativity of (S, T), (S, J), (T, J), it can be shown that

$$S\,p = T\,p = J\,p = ST\,p = p. \qquad (12)$$

Now we need to show that $A\,p = S\,p$ ($B\,p = T\,p$) also remains a common fixed point of both the pairs (AB, I) and (ST, J).
For this put $x = p$ and $y = p$ in (1) and using (11) and (12), we get

$$\phi\,(F_{ABp,\,STp}(kt)\,,\,F_{Ip,\,Jp}(t)\,,\,F_{ABp,\,Ip}(t)\,,\,F_{STp,\,Jp}(kt)) \geq 0$$

that is

$$\phi\,(F_{Ap,\,Sp}(kt)\,,\,F_{Ap,\,Sp}(t)\,,\,F_{Ap,\,Ap}(t)\,,\,F_{Sp,\,Sp}(kt)) \geq 0$$

As ϕ is non decreasing in first argument, we have

$$\phi\,(F_{Ap,\,Sp}(t)\,,\,F_{Ap,\,Sp}(t)\,,1\,,\,1) \geq 0$$

Using (ii), we obtain

$F_{Ap,\,Sp}(t) \geq 1$ for all $t > 0$, which gives $F_{Ap,\,Sp}(t) = 1$, that is $A\,p = S\,p$. Similarly it can be shown that $B\,p = T\,p$.
Thus p is the unique common fixed point of A, B, S, T, I and J. This completes the proof of our theorem.
Since semi-compatibility implies weak compatibility we have the following

Corollary 3.2. Let A, B, S, T, I and J be self-maps of a complete Menger space (X, F, Min) satisfying the conditions (a), (1) and (2) of the above theorem and the pairs (AB, I) and (ST, J) are semi-compatible and one of the pair (AB, I) or

(ST, J) is reciprocally continuous .

Then AB , ST , I and J have a unique common fixed point . Furthermore if the pairs (A , B) , (A , I) , (B , I) , (S , T) , (S , J) and (T , J) are commuting mapping then A , B , S , T , I and J have a unique common fixed point .
If we take B = T = identity map in theorem 3.1 then

Corollary 3.3. Let A, S, I and J be self-mappings of a complete Menger space (X , F , Min) such that

(a) $A(X) \subset J(X)$ and $S(X) \subset I(X)$
(b) (A , I) , (S , J) are semi-compatible commuting pair of maps .
(c) the pair (A , I) or (S , J) is reciprocally continuous .

For some $\phi \in \Phi$, there exists $k \in (0 , 1)$ such that for all x , y $\in$ X and $t > 0$,

$\phi(F_{Ax,Sy}(kt), F_{Ix,Jy}(t), F_{Ax,Ix}(t), F_{Sy,Jy}(kt)) \geq 0$ (13)

$\phi(F_{Ax,Sy}(kt), F_{Ix,Jy}(t), F_{Ax,Ix}(kt), F_{Sy,Jy}(t)) \geq 0$ (14)

Then A , S , I and J have a unique common fixed point .
This theorem proves that the theorem of Singh holds even the pairs are reciprocally continuous .
On taking I = identity mapping in theorem 3.1 , we have following result for 5 self-maps

Corollary 3.4. Let A, B, S, T and J be self-mappings of a complete Menger space (X , F , Min) such that

(a) $AB(X) \cap ST(X) \subset J(X)$
(b) (ST, J) is weak compatible**.**

For some $\phi \in \Phi$, there exists $k \in (0 , 1)$ such that for all x , y $\in$ X and $t > 0$,

$\phi(F_{ABx,STy}(kt), F_{x,Jy}(t), F_{ABx,x}(t), F_{STy,Jy}(kt)) \geq$ (15)

$\phi(F_{ABx,STy}(kt), F_{x,Jy}(t), F_{ABx,x}(kt), F_{STy,Jy}(t)) \geq 0$ (16)

Then AB , ST and J have a unique common fixed point . Furthermore , if the pairs (A , B) , (S , T) , (S , J) and (T , J) are commuting mapping then A , B , S , T and J have a unique common fixed point .
If we take B = T = I = J = identity mapping in theorem 3.1 then the conditions (a) , (b) and (c) are satisfied trivially and we get the following result

Corollary 3.5. Let A and S be self-mappings of a complete Menger space (X , F , Min) such that for some $\phi \in \Phi$, there exists $k \in (0 , 1)$ such that for all x , y $\in$ X and $t > 0$,

$\phi(F_{Ax,Sy}(kt), F_{x,y}(t), F_{Ax,x}(t), F_{Sy,y}(kt)) \geq 0$ (17)

$\phi(F_{Ax,Sy}(kt), F_{x,y}(t), F_{Ax,x}(kt), F_{Sy,y}(t)) \geq 0$ (18)

Then A and S have a unique common fixed point in X .

Theorem 3.6. Let A , B , S , T , I and J be self-maps on a complete Menger space (X , F , Min) satisfying conditions (a), (c), (1) and (2) of theorem 3.1 and (AB , I) is compatible and (ST , J) is weak compatible . Then AB , ST , I and J have a unique common fixed point. Furthermore if the pairs (A , B) , (A , I) , (B , I) , (S , T) , (S , J) and (T , J) are commuting mapping then A , B , S , T , I and J have a unique common fixed point .

Proof. As in the proof of theorem 3.1 , the sequence $\{y_n\}$ converges to $p \in X$ and (6) and (7) are satisfied.
The reciprocal continuity of the pair (AB , I) gives

$ABIx_{2n} \to ABp$ and $IABx_{2n} \to Ip$

The compatibility of the pair (AB , I) gives

$\lim_{n\to\infty} ABIx_{2n}$ = ABp = $\lim_{n\to\infty} IABx_{2n}$

From the uniqueness of the limit in a Menger space , we obtain that ABp = Ip
Step 1. By putting x = AB x_{2n} , y = x_{2n+1} in (1) , we obtain

$\phi(F_{ABABx_{2n}, STx_{2n+1}}(kt), F_{IABx_{2n}, Jx_{2n+1}}(t), F_{ABABx_{2n}, IABx_{2n}}(t), F_{STx_{2n+1}, Jx_{2n+1}}(kt)) \geq 0$

Letting n tends to infinity and using (6) , (7) and (8) we get

$\phi(F_{ABp,p}(kt), F_{ABp,p}(t), F_{ABp,ABp}(t), F_{p,p}(kt)) \geq 0$

As ϕ is non decreasing in first argument, we have

$\phi(F_{ABp,p}(t), F_{ABp,p}(t), 1, 1) \geq 0$

Using (ii), we have $F_{ABp,p}(t) \geq 1$ for all $t > 0$, which gives $F_{ABp,p}(t) = 1$, that is

$$Ip = p = ABp$$

Step 2. As $AB(X) \subset J(X)$, there exists $u \in X$ such that $AB\,p = I\,p = p = J\,u$.
Putting $x = x_{2n}$, $y = u$ in (1) we obtain that

$\phi(F_{ABx_{2n},STu}(kt), F_{Ix_{2n},Ju}(t), F_{ABx_{2n},Ix_{2n}}(t), F_{STu,Ju}(kt)) \geq 0$

Letting n tends to infinity and using (6) and (7) we get

$\phi(F_{p,STu}(kt), 1, 1, F_{STu,p}(kt)) \geq 0$

Using (i), we have $F_{p,STu}(kt) \geq 1$ for all $t > 0$, which gives $F_{p,STu}(kt) = 1$.
Thus $p = ST\,u$. Therefore, $ST\,u = J\,u = p$. Since (ST, J) is weak compatible, we get
$JST\,u = STJ\,u$, that is,
$ST\,p = J\,p$

Step 3. By putting $x = p$, $y = p$ in (1) and using (9) and (10), we obtain

$\phi(F_{ABp,STp}(kt), F_{Ip,Jp}(t), F_{ABp,Ip}(t), F_{STp,Jp}(kt)) \geq 0$ that is,

$\phi(F_{ABp,STp}(kt), F_{ABp,STp}(t), 1, 1) \geq 0$.

As ϕ is non decreasing in first argument, we have

$\phi(F_{ABp,STp}(t), F_{ABp,STp}(t), 1, 1) \geq 0$

Using (ii), we have $F_{ABp,STp}(t) \geq 1$ for all $t > 0$, which gives $F_{ABp,STp}(t) = 1$

Thus $AB\,p = ST\,p$.

Therefore, $p = AB\,p = ST\,p = I\,p = J\,p$, that is p is a common fixed point of AB, ST, I and J.
The rest of the proof is same as in theorem 3.1.
Since compatibility implies weak compatibility, we have the following

Corollary 3.7. Let A, B, S, T, I and J be self-maps on a complete Menger space (X, F, Min) satisfying conditions (a), (1) and (2) of theorem 3.1 and the pairs (AB, I) and (ST, J) are compatible and one of the pair (AB, I) or (ST, J) is reciprocally continuous. Then AB, ST, I and J have a unique common fixed point in X. Furthermore if the pairs(A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mapping then A, B, S, T, I and J have a unique common fixed point.

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