Volume 6, Issue 11, November 2016 ISSN: 2277 128X
International Journal of Advanced Research in Computer Science and Software Engineering
Research Paper
Available online at: www.ijarcsse.com

On The Massey Theorem in E_ν^{n+1}

Ali Görgülü, Cumali Ekici*
Department of Mathematics-Computer, Eskişehir Osmangazi University, Turkey

Abstract— ***In this paper, firstly, we define the generalized $(k+1)$-dimensional semi-ruled surface whose directional surface is a semi-subspace in the semi-Euclidean space E_ν^{n+1}. Then we investigate the sufficient and necessary conditions for these surfaces to be totally developable. In addition, we give the generalization of Massey theorem, which is well-known for the ruled surfaces defined in 3-dimensional Euclidean space, for the $(k+1)$-dimensional ruled surfaces in the semi-Euclidean space E_ν^{n+1}.***

Keywords— Ruled surface, Massey Theorem, Semi-Euclidean space.

I. INTRODUCTION

A special surface generated by a continuously moving of a straight line is a ruled surface. Since ruled surfaces have the most important positions and applications in the study of design problems in spatial mechanisms and physics, kinematics and computer aided design (CAD), these surfaces are one of the most important topics in differential geometry. There are a lot of studies related to two-dimensional or $(k+1)$-dimensional ruled surfaces and their properties in n-dimensional Euclidean space [5, 6, 7, 8, 9, 14]. Some geometers have studied generalized ruled surfaces and obtained many interesting results in both Euclidean and semi-Euclidean spaces [4, 10, 11, 12, 13, 15]. Further, the basic definitions and theorems related to the Minkowski space R_1^n and semi-Euclidean spaces are also studied in many papers [1, 2, 4, 10-13, 15].

We will assume throughout the paper that all manifolds, maps, vector fields, etc... are differentiable of class C^∞. Let R^n be the n-dimensional vector space. The following symmetric, bilinear and non-degenerate metric tensor is called the Lorentzian metric on R^n:

$$<X,Y> = \sum_{i=1}^{n-1} x_i y_i - x_n y_n, \quad X=(x_1,x_2,...x_n), \quad Y=(y_1,y_2,...,y_n)$$

E_ν^{n+1} semi-Euclidean space is an Euclidean space provided with the metric tensor

$$\langle X_p, Y_p \rangle = \sum_{i=1}^{n-\nu} x_i y_i - \sum_{i=n-\nu+1}^{n+1} x_i y_i$$

where $X=(x_1,x_2,...,x_{n+1})$ and $Y=(y_1,y_2,...,y_{n+1})$. Particularly, if $\nu=0$ then E_0^{n+1} is called Euclidean $(n+1)$-space, if $\nu=1$ $(n\geq 2)$ then E_1^{n+1} is called as Minkowski $(n+1)$-space (O'Neill, 1983).

II. PRELIMINARIES

Let M be a semi-Euclidean submanifold of E_ν^{n+1} and $\bar{\nabla}$ be a Levi-Civita connection of E_ν^{n+1} and ∇ be Levi-Civita connection of M. If $\forall X,Y \in \chi(M)$ and P is second fundamental form of M, then we have the Gauss equation

$$\bar{\nabla}_X Y = \nabla_X Y + P(X,Y) \tag{1}$$

Let ξ be a unit normal vector of M. Then the Weingarten equation is

$$\bar{\nabla}_X \xi = -A_\xi(X) + \nabla_X^\perp \xi \tag{2}$$

where A_ξ determines a self-adjoint linear map at each point of T_pM and $\nabla^\perp$ is a metric connection [3]. We note that in this paper, A_ξ will be used for the linear map and the corresponding matrix of this linear map. Then we have

$$\langle P(X,Y), x\rangle = \langle A_x(X), Y\rangle \tag{3}$$

and

$$P(X,Y) = \sum_{j=1}^{n-m} \langle A_{x_j}(X), Y\rangle x_j \tag{4}$$

For every $X_i \in \chi(M), i = 1,2,3,4$ the $4th$ order covariant tensor field

$$R(X_1,X_2,X_3,X_4) = \langle X_1, R(X_3,X_4)X_2\rangle \tag{5}$$

is called the Riemannian curvature tensor field and its value at a point $P \in M,$ is called Riemannian curvature of M at $P,$ where M is an m-dimensional semi-Riemann submanifold in E_n^{n+1} [1].

Let M be an m-dimensional semi-Riemannian manifold. A 2-dimensional subspace of the tangent space T_pM of M at P is called tangent plane of M at P and is denoted by Π. For all $X_p, Y_p \in \Pi$, the real valued function K defined by

$$K(X_p,Y_p) = \frac{\langle R(X_p,Y_p)Y_p, X_p\rangle}{\langle X_p,X_p\rangle\langle Y_p,Y_p\rangle - \langle X_p,Y_p\rangle^2} \tag{6}$$

is called the sectional curvature function of M at the point P [3].

III. $(k+1)$-DIMENSIONAL SEMI-RULED SURFACES IN E_n^{n+1}

Let α be the smooth curve

$$\alpha : I \to E_n^{n+1}$$
$$t \to \alpha(t) = (\alpha_1(t),...,\alpha_{n+1}(t))$$

in the $(n+1)$-dimensional semi-Euclidean space E_n^{n+1} where $\{0\} \subset I \subset R.$ Let $\{e_1(t), e_2(t),..., e_k(t)\}$ be an ortonormal vector system defined at each point $\alpha(t)$ of the curve α. This system spans a subspace of the tangent space $T_{E_n^{n+1}}(\alpha(t))$ at $\alpha(t) \in E_n^{n+1}.$ If this space is shown by $E_{k,m}(t),$ then, it is a k-dimensional subspace of the form

$$E_{k,m}(t) = Sp\{e_1(t), e_2(t),..., e_k(t)\} \subset E_n^{n+1}, \quad 0 \le m \le n$$

$E_{k,m}(t),\ m \ge 1,$ is called a semi-subspace and satisfied

$$\langle e_i(t), e_j(t)\rangle = \varepsilon_i \delta_{ij}, \quad \varepsilon_i = \begin{cases} +1 & , 1 \le i \le k-m \\ -1 & , k-m+1 \le i \le k \end{cases}$$

For $m \ge 1,$ there are m time-like vectors in the semi-subspace $E_{k,m}(t).$ Since there is no time-like vector on $E_{k,0}(t)$ for $m = 0$, then $E_{k,0}(t) = E_k(t)$ and it is an Euclidean subspace. If $m = 1,$ then, there is one time-like vector so $E_{k,1}(t)$ is a time-like subspace.

Throughout this paper, we assume that $E_{k,m}(t), m \ge 1,$ is a semi-subspace.

Definition 1 *While the semi-subspace* $E_{k,m}(t)$ *moves along a curve* α *in* $E_n^{n+1},$ *it forms a* $(k+1)$*-dimensional surface. This surface is called the* $(k+1)$ *-dimensional generalized semi-ruled surface in the* $(n+1)$ *-dimensional semi-Euclidean space* E_n^{n+1} *and is denoted by* M^*.

Definition 2 $E_{k,m}(t)$ *is called the generating space at* $\alpha(t)$ *of the semi-ruled surface* $M^*,$ *and the curve* α *is called the base curve of* M^* .

Let M^* be a $(k+1)$-dimensional generalized semi-ruled surface in E_n^{n+1} and

$$E_{k,m}(t) = Sp\{e_1(t), e_2(t),..., e_k(t)\} \subset E_n^{n+1}, \quad 0 \le m \le n$$

be the generating space α be the base curve parametrized by the arc-lenght. Then M^* can be expressed by the parametric equation

$$f(t,u_1,u_2,...,u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t) \ , \ (t,u_1,u_2,...,u_k) \in I \times \mathbf{R}^k \tag{7}$$

Let M^* be a $(k+1)$-dimensional semi-ruled surface in the semi-Euclidean space E_n^{n+1}, let $\{e_1(t), e_2(t), ..., e_k(t)\}$ be the natural companion basis of the generating space $E_{k,m}(t)$ and $e_0 = f_*(\frac{\partial}{\partial t})$ be a vector field such that $\{e_0, e_1, ..., e_k\}$ is an orthonormal basis of $\chi(M^*)$. In addition, suppose that the vector field system $\{x_{k+1}, x_{k+2}, ..., x_n\}$ is an orthonormal basis of $T^{\wedge}_{M^*}(P)$ at $P \in M^*$. Then

$$\{e_0, e_1, ..., e_k, x_{k+1}, x_{k+2}, ..., x_n\}$$

is an orthonormal basis of $T_{E_n^{n+1}}(p)$ at $P \in M^*$. Thus, the derivative equations can be written in the form

$$\bar{\nabla}_{e_r} x_j = \sum_{i=0}^{k} a_{ri}^j e_i + \sum_{i=k+1}^{n} b_{ri}^j x_i \ , \ 0 \le r \le k \text{ ve } k+1 \le j \le n \tag{8}$$

Hence, the Weingarten equation can be written as

$$\bar{\nabla}_{e_r} x_j = -A_{x_j}(e_r) + \nabla^{\wedge}_{e_r} x_j \quad , \quad k+1 \le j \le n \tag{9}$$

Definition 3 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_n^{n+1} and P be the second fundamental form of M^*. If*

$$P(X, X) = 0$$

for all $X \in \chi(M^*)$, then X is called asymptotic vector field on M^*.

Theorem 4 *In [4], let M^* be $(k+1)$-dimensional semi-ruled surface in E_n^{n+1} and $\{e_0, e_1, e_2, ..., e_k\}$ be an orthonormal vector system at a neighborhood of a point $p \in M^*$. The Riemann curvature of the section which is spanned by the vectors $(e_i)|_p$, $1 \le i \le k$, and $(e_0)|_p$ of M^* is*

$$K(e_i, e_0) = -\varepsilon_{0i} \langle \bar{\nabla}_{e_i} e_0, \bar{\nabla}_{e_i} e_0 \rangle \ , \ \varepsilon_{0i} = \varepsilon_0 \varepsilon_i. \tag{10}$$

Theorem 5 *Let M^*, be a $(k+1)$-dimensional semi-ruled surface in E_n^{n+1} and $\{e_1, e_2, ..., e_k\}$ be an orthonormal base field of the generating $E_{k,m}(t)$. Then the lines, which correspond to $e_1, e_2, ..., e_k$ are the asymptotics and geodesics of M^*.*

Proof. Since the lines, correspondings to the orthonormal base field vectors $e_1, e_2, ..., e_k$ of the generating space $E_{k,m}(t)$, are geodesics of E_n^{n+1}, we have

$$\bar{\nabla}_{e_i} e_i = 0 \quad , 1 \le i \le k$$

By substituting this equation into the Gauss equation, we get

$$\nabla_{e_i} e_i = -P(e_i, e_i)$$

Since $\nabla_{e_i} e_i \in \chi(M^*)$ and $P(e_i, e_i) \in \chi^{\wedge}(M^*)$ we have

$$\nabla_{e_i} e_i = 0 \text{ ve } P(e_i, e_i) = 0$$

Therefore the lines,corresponding to $e_1, e_2, ..., e_k$, are the asymptotics and geodesics of M^*.

Definition 6 *Suppose that $\{e_0, e_1, e_2, ..., e_k\}$ is an orthonormal base field of a $(k+1)$-ruled surface M^*, where e_0 is the unit tangent vector of the orthogonal trajectories of the generating space $E_{k,m}(t)$. If*

$$\text{rank}\left[e_0, e_1, ..., e_k, \bar{\nabla}_{e_0} e_1, \bar{\nabla}_{e_0} e_2, ..., \bar{\nabla}_{e_0} e_k\right] = 2k - m$$

at each point P of M^*, then the M is called as $m-$ developable. If $m = -1$, then M is called as non-developable. If $m = k-1$, M is called as total developable.

Now we would like to give a theorem, which is characterizing the total developability of M^*.

Theorem 7 *Let M^* a $(k+1)$- dimensional semi-ruled surface in E_n^{n+1} and $\{e_0, e_1, ..., e_k\}$ be an orthonormal base field of M^*. Then M^* total developable iff*

$$\bar{\nabla}_{e_i} e_0 = 0 \ , \ 1 \le i \le k. \tag{11}$$

Proof. $(\Rightarrow)$: Assume that M is total developable. Then we have

$$\operatorname{rank}\left[e_0, e_1, \ldots, e_k, \bar{\nabla}_{e_0} e_1, \bar{\nabla}_{e_0} e_2, \ldots, \bar{\nabla}_{e_0} e_k\right] = k+1$$

at each point of M^*. Moreover, from Gauss equation we know that

$$\bar{\nabla}_{e_0} e_i = \nabla_{e_0} e_i + P(e_0, e_i) \quad , \; 1 \le i \le k$$

Since M^* is total developable and the system $\{e_0, e_1, \ldots, e_k\}$ is linearly independent, $\bar{\nabla}_{e_0} e_i$ has no component in the normal bundle $\chi^{\perp}(M^*)$ that is $P(e_i, e_i) = 0.$ Therefore we have

$$\bar{\nabla}_{e_0} e_i = \nabla_{e_0} e_i \quad , \; 1 \le i \le k \tag{12}$$

Similarly, from the Gauss equation and since P is symmetric we find

$$\bar{\nabla}_{e_i} e_0 = \nabla_{e_i} e_0 \;\text{ and }\; \langle \bar{\nabla}_{e_i} e_0, e_0 \rangle = 0 \;, \; 1 \le i \le k \tag{13}$$

and by (13)

$$\langle \bar{\nabla}_{e_i} e_0, \xi_j \rangle = 0 \quad , \; 1 \le i \le k, \;\; k+1 \le j \le n \tag{14}$$

since

$$\langle \bar{\nabla}_{e_i} e_0, e_r \rangle = -\langle e_0, \bar{\nabla}_{e_i} e_r \rangle$$

and $\bar{\nabla}_{e_i} e_r = 0, \; 1 \le i, r \le k.$ We observe that

$$\langle \bar{\nabla}_{e_i} e_0, e_r \rangle = 0 \quad , \; 1 \le i, r \le k \tag{15}$$

Therefore, the equations (13), (14) and (15) show that $\bar{\nabla}_{e_i} e_0$ has no tangential and normal component. Therefore

$$\bar{\nabla}_{e_i} e_0 = 0 \quad , \; 1 \le i, r \le k$$

$\Leftarrow$: Assume that $\bar{\nabla}_{e_i} e_0 = 0, \; 1 \le i \le k.$ Then we will show that

$$\bar{\nabla}_{e_0} e_i \in Sp\{e_0, e_1, \ldots, e_k\}$$

Since $P(e_0, e_i) = P(e_i, e_0)$, from the Gauss equation and the hypothesis we have $P(e_i, e_0) = 0.$ Substituting this into the Gauss equation gives

$$\bar{\nabla}_{e_0} e_i = \nabla_{e_0} e_i + P(e_0, e_i),$$

and we obtain

$$\bar{\nabla}_{e_0} e_i = \nabla_{e_0} e_i$$

Therefore,

$$\bar{\nabla}_{e_0} e_i \in Sp\{e_0, e_1, \ldots, e_k\}$$

that is, every $\bar{\nabla}_{e_0} e_i$ is linearly dependent with the orthonormal base field $\{e_0, e_1, \ldots, e_k\}$. Thus we observe that

$$\operatorname{rank}\left[e_0, e_1, \ldots, e_k, \bar{\nabla}_{e_0} e_1, \bar{\nabla}_{e_0} e_2, \ldots, \bar{\nabla}_{e_0} e_k\right] = k+1$$

That is, M^* is total developable.

For the orthonormal base $\{e_1, \ldots, e_k\}$ of $E_{k,\mu}(t)$

$$\bar{\nabla}_{e_i} e_j = 0, \; 1 \le i, j \le k \tag{16}$$

Substituting this result into the Gauss equation given by (1) implies

$$P(e_i, e_j) = 0 \;, 1 \le i, j \le k \tag{17}$$

since $\nabla_{e_i} e_j \in \chi(M^*)$ and $P(e_i, e_j) \in \chi^{\perp}(M^*), \;\; 1 \le i, j \le k.$ Moreover

$$\bar{\nabla}_{e_i} e_0 \perp e_j \;\text{ and }\; \bar{\nabla}_{e_i} e_0 \perp e_0$$

because

$$\langle e_i(t), e_j(t) \rangle = \varepsilon_i \delta_{ij}, \;\; \varepsilon_i = \begin{cases} +1 & , \; 1 \le i \le k-\mu \\ -1 & , \; k-\mu+1 \le i \le k \end{cases} \;, \; \mu \le n$$

and $\langle e_0, e_j \rangle = 0, \; \langle e_0, e_0 \rangle = \varepsilon_0.$ So $\bar{\nabla}_{e_i} e_0 \in \chi^{\perp}(M^*).$ Then

$$\bar{\nabla}_{e_i} e_0 = P(e_i, e_0) \;, \; 1 \le i \le k. \tag{18}$$

On the other hand, covariant derivative with respect to e_r of the equation $\langle e_h, \xi_j \rangle = 0$ gives

$$\langle \bar{D}_{e_r} e_h, x_j\rangle = -\langle \bar{D}_{e_r} x_j, e_h\rangle$$

Then

$$\varepsilon_h a^j_{rh} = -\langle \bar{D}_{e_r} e_h, x_j\rangle = \langle \bar{D}_{e_r} x_j, e_h\rangle$$

and since the vectors e_r are parallel vectors in E^{n+1}_n we have $\bar{D}_{e_r} e_h = 0$. So

$$\varepsilon_h a^j_{rh} = 0 \text{ and } a^j_{rh} = 0. \quad (19)$$

Therefore the matrix representation of the linear map A_{x_j} is

$$A_{x_j} = -\begin{bmatrix} a^j_{00} & a^j_{01} & a^j_{02} & \cdots & a^j_{0k} \\ a^j_{10} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a^j_{k0} & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} \quad (20)$$

By the derivative equations (8) it be written that

$$\langle \bar{D}_{e_r} x_j, e_h\rangle = \varepsilon_h a^j_{rh}, \ \langle e_i, e_h\rangle = \varepsilon_h \delta_{ih}, \ 0 \leq r, h \leq k$$

and

$$\langle \bar{D}_{e_r} x_j, e_0\rangle = \varepsilon_0 a^j_{r0}$$

Considering $\langle P(X,Y), x\rangle = \langle A_x(X), Y\rangle$ and the Weingarten equation gives

$$\begin{aligned} \langle P(e_r, e_h), x_j\rangle &= \langle A_{x_j}(e_r), e_h\rangle \\ &= \langle -\sum_{i=0}^{k} a^j_{ri} e_i, e_h\rangle \\ &= -\varepsilon_h a^j_{rh}, \quad 0 \leq h \leq k \end{aligned}$$

For $r = 0$ we have

$$\langle P(e_0, e_h), x_j\rangle = \langle A_{x_j}(e_0), e_h\rangle = -\varepsilon_h a^j_{0h} \quad (21)$$

On the other hand

$$\langle P(e_h, e_0), x_j\rangle = \langle A_{x_j}(e_h), e_0\rangle = -\varepsilon_0 a^j_{h0} \quad (22)$$

Moreover, since the second fundamental form is symmetric we have

$$P(e_h, e_0) = P(e_0, e_h)$$

By considering this equation we get

$$\begin{aligned} \varepsilon_0 a^j_{h0} &= -\langle P(e_h, e_0), x_j\rangle = -\langle A_{x_j}(e_h), e_0\rangle \\ &= -\langle P(e_0, e_h), x_j\rangle = -\langle A_{x_j}(e_0), e_h\rangle \\ &= \varepsilon_h a^j_{0h} \end{aligned}$$

that is, for the components of A_{x_j}

$$a^j_{h0} = \varepsilon_0 \varepsilon_h a^j_{0h}$$

By substituting this result into (20) we have

$$A_{x_j} = -\begin{bmatrix} a^j_{00} & a^j_{01} & a^j_{02} & \cdots & a^j_{0k} \\ \varepsilon_{01} a^j_{10} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \varepsilon_{0k} a^j_{k0} & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} \quad (23)$$

where $\varepsilon_{0i} = \varepsilon_0 . \varepsilon_i$, $\langle e_0, e_0\rangle = \varepsilon_0$.

Therefore these results can be given for the matrix A_{x_j} defined with the Weingarten derivative equations.

Corollary Let M^* be a $(k+1)$-dimensional semi-ruled surface in E^{n+1}_n. The matrix in (23) which corresponds to the shape operator A_{x_j} of M^* is symmetric or anti-symmetric depending on the index.

Corollary Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_n^{n+1} and A_{x_j} be the shape operator defined for the unit normal direction x_j, $k+1 \le j \le n.$ Then for $k \ge 2$

$$\det A_{x_j} = 0.$$

Corollary Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_n^{n+1}. The Lipschitz-Killing curvature on every point and every normal direction of M^* is zero for $k \ge 2$ [4].

It can be written by the Weingarten derivative equations that

$$\langle \overline{D}_{e_i} x_j, e_0 \rangle = e_0 a_{i0}^j \quad , \ 1 \le i \le k$$

Moreover, since $\langle x_j, e_0 \rangle = 0$ we obtain

$$e_i[\langle x_j, e_0 \rangle] = \langle \overline{D}_{e_i} x_j, e_0 \rangle + \langle x_j, \overline{D}_{e_i} e_0 \rangle = 0.$$

So

$$\langle \overline{D}_{e_i} x_j, e_0 \rangle = -\langle x_j, \overline{D}_{e_i} e_0 \rangle.$$

Left hand side of this expression is $e_0 a_{i0}^j$. Therefore, since

$$\langle x_j, \overline{D}_{e_i} e_0 \rangle = -e_0 a_{i0}^j$$

or $a_{i0}^j = e_{0i} a_{0i}^j = e_0 e_i a_{0i}^j$ we get

$$\langle x_j, \overline{D}_{e_i} e_0 \rangle = -e_i a_{0i}^j.$$

In addition, since $\langle \overline{D}_X x, Y \rangle = \langle A_x(X), Y \rangle$ and $\langle P(X,Y), x \rangle = \langle A_x(X), Y \rangle$ it can be written that

$$\langle \overline{D}_{e_i} e_0, x_j \rangle = \langle A_{x_j}(e_i), e_0 \rangle = -e_i a_{0i}^j$$

and

$$\langle A_{x_j}(e_i), e_0 \rangle = \langle P(e_i, e_0), x_j \rangle$$

Then

$$\langle P(e_i, e_0), x_j \rangle = -e_i a_{0i}^j.$$

Substituting this result into $P(X,Y) = \sum_{j=k+1}^{n} \langle A_{x_j}(X), Y \rangle x_j$ gives

$$P(e_i, e_0) = \sum_{j=k+1}^{n} \langle P(e_i, e_0), x_j \rangle x_j = -\sum_{j=k+1}^{n} e_i a_{0i}^j x_j \tag{24}$$

By combining (18) and $P(e_i, e_0)$ we can obtain

$$\overline{D}_{e_i} e_0 = -\sum_{j=k+1}^{n} e_i a_{0i}^j x_j \quad , \ 1 \le i \le k. \tag{25}$$

Theorem 8 *Let* M^* *be a* $(k+1)$*-dimensional semi-ruled surface in* E_n^{n+1} *and* $\{e_0, e_1, e_2, ..., e_k\}$ *be an orthonormal vector system on a neighborhood of the point* $p \in M^*$. *Then the Riemann curvature of the 2-dimensional section of* M^* *spanned by the vectors* $(e_i)|_p$, $1 \le i \le k,$ *and* $(e_0)|_p$ *is*

$$K(e_i, e_0) = -e_{0i} \langle \overline{D}_{e_i} e_0, \overline{D}_{e_i} e_0 \rangle , \ e_{0i} = e_0 e_i. \tag{26}$$

Proof. Suppose that the system of vector fields $\{x_{k+1}, x_{k+2}, ..., x_n\}$ is an orthonormal basis of the space $T_{M^*}^{\perp}(p)$ at the point $P \in M^*$. Then an orthonormal basis of the space $T_{E_n^{n+1}}(p)$ at the point $P \in M^*$ is

$$\{e_0, e_1, ..., e_k, x_{k+1}, x_{k+2}, ..., x_n\}$$

Writing the covariant derivatives $\overline{D}_{e_r} x_j$ of x_j , $k+1 \le j \le n,$ with respect to e_r , $0 \le r \le k,$ in terms of these basis vectors gives

$$\overline{D}_{e_r} x_j = \sum_{i=0}^{k} a_{ri}^j e_i + \sum_{i=k+1}^{n} b_{ri}^j x_i \quad , \ 0 \le r \le k \text{ ve } k+1 \le j \le n.$$

One can find the Weingarten equation

$$\overline{D}_{e_r} x_j = -A_{x_j}(e_r) + D^{\perp}_{e_r} x_j \quad , \quad k+1 \le j \le n \tag{27}$$

for the basis vectors. By writing $\overline{D}_{e_r} x_j$ for $0 \le r \le k$ and $k+1 \le j \le n$ we have

$$\begin{aligned} \overline{D}_{e_0} x_j &= a^j_{00} e_0 + \sum_{i=1}^{k} a^j_{0i} e_i + \sum_{i=k+1}^{n} b^j_{0i} x_i \\ \overline{D}_{e_1} x_j &= a^j_{10} e_0 + \sum_{i=1}^{k} a^j_{1i} e_i + \sum_{i=k+1}^{n} b^j_{1i} x_i \\ &\vdots \\ \overline{D}_{e_k} x_j &= a^j_{k0} e_0 + \sum_{i=1}^{k} a^j_{ki} e_i + \sum_{i=k+1}^{n} b^j_{ki} x_i \end{aligned} \tag{28}$$

These equations are called the Weingarten derivative equations. By comparing the equations (27) and (28) it can be written that

$$-A_{x_j}(e_r) = \sum_{i=0}^{k} a^j_{ri} e_i \ , \quad 0 \le r \le k$$

since $A_{x_j}(e_r) \in \chi(M^*)$. Representing the matrix of the linear mapping A_{x_j} by the same notation gives

$$A_{x_j} = -\begin{bmatrix} a^j_{00} & a^j_{01} & a^j_{02} & \cdots & a^j_{0k} \\ a^j_{10} & a^j_{11} & a^j_{12} & \cdots & a^j_{1k} \\ \vdots & \vdots & \vdots & & \vdots \\ a^j_{k0} & a^j_{k1} & a^j_{k2} & \cdots & a^j_{kk} \end{bmatrix}_{(k+1)\times(k+1)} \quad , \ k+1 \le j \le n \tag{29}$$

by the Weingarten equations above. In addition since the direction space $E_{k,m}(t)$ is semi-subspace we have

$$\langle e_i, e_h \rangle = \varepsilon_i \delta_{ih} \ , \quad 1 \le i, h \le k,$$

where

$$\varepsilon_i = \begin{cases} +1 & , \ 1 \le i \le k-m \\ -1 & , \ k-m+1 \le i \le k \end{cases} \quad \text{and } m \le n$$

for the orthonormal basis of $E_{k,m}(t)$. Therefore, inner product of both sides of the derivative equations (28) with e_h gives

$$\langle \overline{D}_{e_r} x_j, e_h \rangle = \langle \sum_{i=0}^{k} a^j_{ri} e_i + \sum_{i=k+1}^{n} b^j_{ri} x_i, e_h \rangle = \varepsilon_h a^j_{ih}.$$

IV. THE MASSEY THEOREM FOR THE $(k+1)$-DIMENSIONAL IN E^{n+1}_n

Consider a $(k+1)$- dimensional ruled surface M^* in E^{n+1}_n and the orthonormal base field $\{e_1, e_2, ..., e_k\}$ of the generating space $E_{k,m}(t)$. Then the orthonormal base field $\{e_0, e_1, ..., e_k\}$ of the tangential bundle of M at each point P of M^* and the orthonormal base field $\{x_{k+1}, x_{k+2}, ..., x_n\}$ of the normal bundle of M^* at each point P of M^* constitute an orthonormal base field of E^{n+1}_n at each point P of E^{n+1}_n.

Moreover, we can give covariant derivative equations of the orthonormal base field $\{e_0, e_1, ..., e_k, x_{k+1}, x_{k+2}, ..., x_n\}$ of E^{n+1}_n

$$\begin{bmatrix} \overline{D}_{e_0} e_0 \\ \overline{D}_{e_0} e_1 \\ \vdots \\ \overline{D}_{e_0} e_k \\ \overline{D}_{e_0} x_{k+1} \\ \vdots \\ \overline{D}_{e_0} x_n \end{bmatrix} = \begin{bmatrix} 0 & c_{01} & \cdots & c_{0k} & \cdots & c_{0n} \\ -\varepsilon_{01} c_{01} & 0 & \cdots & c_{1k} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_{0k} c_{0k} & -\varepsilon_{1k} c_{1k} & \cdots & 0 & \cdots & c_{kn} \\ -\varepsilon_{0(k+1)} c_{0(k+1)} & [illegible] & \cdots & -\varepsilon c & \cdots & c_{(k+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_{0n} c_{0n} & -\varepsilon_{1n} c_{1n} & \cdots & -\varepsilon_{kn} c_{kn} & \cdots & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_k \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix} \tag{30}$$

where $e_i.e_j = e_{ij}$, $r = 1(k+1)$, $l = k(k+1)$ and $c_{ij} \in \mathbf{R}$.

Now, we would like to generalize the Massey's Theorem, which is well-known for the ruled surface in E^n [10], [14], to the $(k+1)$- dimensional semi-ruled surfaces in the Euclidean space E_n^{n+1}.

Theorem 9 *(Massey's Theorem): Let* M^* *be* $(k+1)$ *-dimensional semi-ruled surface in* E_n^{n+1}, $\{e_1, e_2, ..., e_k\}$ *be an orthonormal base field of the generating space* $E_{k,m}(t)$ *and* α *be an orthogonal trajectory of the generating space* $E_{k,m}(t)$. *Then the following propositions are equivalent.*

(i) M^* is total developable.

(ii) The sectional curvature $K(e_i, e_0)$ of M^* is zero for $1 \le i \le k$.

(iii) In the equation (15), $c_{rl} = 0$, $1 \le r \le k$, $k+1 \le l \le n$.

(iv) $A_{x_j}(e_i) = 0$, $1 \le i \le k, k+1 \le j \le n$.

(v) $\bar{D}_{e_0} e_i \in \chi(M^*)$, $1 \le i \le k$.

Proof. $(i) \Rightarrow (ii)$: Assume that M^* is total developable. Then, by (11) and (26) we have

$$\bar{D}_{e_i} e_0 = 0 \quad , 1 \le i \le k$$

and

$$K(e_i, e_0) = -\varepsilon_{0i} \langle \bar{D}_{e_i} e_0, \bar{D}_{e_i} e_0 \rangle \quad , \varepsilon_{0i} = \varepsilon_0.\varepsilon_i, \; 1 \le i \le k$$

In the last equation, by setting $\bar{D}_{e_i} e_0 = 0$ we can obtain

$$K(e_i, e_0) = 0 \; , 1 \le i \le k$$

$(ii) \Rightarrow (iii)$: Let $K(e_i, e_0) = 0$, $1 \le i \le k$. From (11) and the first equation of (28), we get

$$\langle \bar{D}_{e_0} x_j, e_i \rangle = 0 \; , \; 1 \le i \le k, \; k+1 \le j \le n$$

This equation shows that $\bar{D}_{e_0} x_j$ has no component in the directions of $e_1, e_2, ..., e_k$. Hence we have

$$\varepsilon_{rl} c_{rl} = 0 \text{ or } c_{rl} = 0, \; 1 \le r \le k, \; k+1 \le l \le n$$

in the equation (30).

$(iii) \Rightarrow (iv)$: Now, we assume that $c_{rl} = 0$, $1 \le r \le k$, $k+1 \le l \le n$. Then, from the equation (30), we observe that $\bar{D}_{e_0} x_j$ has no component in the directions of $e_1, e_2, ..., e_k$. Therefore, from the first equation of (28), we get

$$a_{0i}^j = 0 \quad , \; 1 \le i \le k, \; k+1 \le j \le n$$

By considering this last equation with the Weingarten equations, we obtain

$$A_{x_j}(e_i) = 0 \quad , \; 1 \le i \le k, \; k+1 \le j \le n$$

$(iv) \Rightarrow (v)$: Let $A_{x_j}(e_i) = 0$, $1 \le i \le k$, $k+1 \le j \le n$. Then, from (28) we have

$$a_{0i}^j = 0 \quad , \; 1 \le i \le k, \; k+1 \le j \le n$$

and $\bar{D}_{e_0} x_j$ has no component in the directions of $e_1, e_2, ..., e_k$, i.e.

$$c_{rl} = 0 \quad , 1 \le r \le k, \; k+1 \le l \le n$$

Since the matrix (30) is anti-symmetric, $\bar{D}_{e_0} e_i$ has no component in the direction of x_j, $k+1 \le j \le n$. Hence $\bar{D}_{e_0} e_i \in \chi(M^*)$, $1 \le i \le k$.

$(v) \Rightarrow (i)$: Let $\bar{D}_{e_0} e_i \in \chi(M^*)$, $1 \le i \le k$. This means that, $\bar{D}_{e_0} e_i \in Sp\{e_0, e_1, ..., e_k\}$. From (11)

$$\text{rank}\left[e_0, e_1, ..., e_k, \bar{D}_{e_0} e_1, \bar{D}_{e_0} e_2, ..., \bar{D}_{e_0} e_k \right] = k+1$$

Therefore, M is total developable, i.e.

$$\bar{D}_{e_i} e_0 = 0 \; , \; 1 \le i \le k$$

This completes the proof.

Special case $n = 0$: Then $m = 0$. Therefore it will be E^{n+1} Euclidean space and $E_{k,0}(t) = E_k(t)$. Then the

$(k+1)$-dimensional generalized ruled surface is obtained in the Euclidean space E^{n+1}. Let M be the $(k+1)$-dimensional generalized ruled surface, $\{e_1, e_2, ..., e_k\}$ be the orthonormal base field of the direction space $E_k(t)$ of M and α be the orthogonal trajectory of the direction space $E_k(t)$. Then these propositions are equivalent.

(i) M is total developable.

(ii) The Riemann curvature $K(e_i, e_0)$, $1 \le i \le k$, of M is zero.

(iii) $c_{rl} = 0$, $1 \le r \le k$, $k+1 \le l \le n$.

(iv) $A_{x_j}(e_i) = 0$, $1 \le i \le k, k+1 \le j \le n$.

(v) $\overline{D}_{e_0} e_i \in \chi(M^*)$, $1 \le i \le k$.

This is the generalization of the classical Massey theorem which is well-known for the ruled surfaces in 3-dimensional Euclidean space E^3 to the $(k+1)$-ruled surfaces in the Euclidean space E^{n+1} [14].

V. CONCLUSIONS

In this paper, firstly, the generalized $(k+1)$-dimensional semi-ruled surface whose directional surface is a semi-subspace in the semi-Euclidean space E_n^{n+1} is defined. Then the sufficient and necessary conditions for these surfaces to be totally developable are investigated. Finally, the generalization of Massey theorem, which is well-known for the ruled surfaces defined in 3-dimensional Euclidean space, is given for the $(k+1)$-dimensional ruled surfaces in the semi-Euclidean space E_n^{n+1}.

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