Volume 5, Issue 9, September 2015 ISSN: 2277 128X
International Journal of Advanced Research in Computer Science and Software Engineering
Research Paper
Available online at: www.ijarcsse.com

A New Lower Bound for the Circumference of Tough Graphs

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***Abstract** – A set of the vertices $A \subseteq V(G)$ and not empty is called an independent set iff every vertex of it is not adjacent to any other. We denote $NC2(G) = min\{|N(u) \cup N(v)|: d(u,v) = 2\}$, where $d(u,v)$ is length of the shortest path joining u and v if G is not complete. Moreover, we define $NC_k(G) = min\{|\cup_{i=1}^{k} N(v_i)|\}$ and $\sigma_k(G) = min\{d(v_1) + d(v_2) + \cdots + d(v_k)\}$ where $\{v_1, v_2, \ldots, v_k\}$ is an independent set of vertices in case there exists the independent set $\{v_1, v_2, \ldots, v_k\}$. A graph G is called to be 1-tough if for every nonempty set $S \subseteq V(G)$, the number component of $G - S$, denoted by $\omega(G - S)$, less than or equal to $|S|$. In this paper, we consider 1-tough graph G with n vertices and $\sigma_3(G) \geq n$. Denote by $c(G)$ the circumference (the length of a longest cycle) of G, Bauer, Fan and Veldman proved that $c(G) \geq min\{n, 2NC2(G)\}$. Vu Dinh Hoa strengthened this result by showing that $c(G) \geq min\{n, 2NC_{\sigma_3 - n+5} + 2$. Our goal in this paper is to prove the conjecture posed by Bauer, Veldman and Fan that $c(G) \geq min\{n, 2NC2(G) + 4\}$.*

***Keywords** – 1-tough graphs, dominating cycles, longest cycle, circumference.*

I. INTRODUCTION

In this paper, we only consider finite undirected graph G with $n \geq 3$ vertices. We use the notation and terms in [3] and denote vertex and edge set of it by $V(G)$and $E(G)$, respectively.

For any vertex $v \in V(G)$, let $N(v)$ and $d(v) = |N(v)|$ denote the set of neighbors and the degree of v, respectively. Moreover, for any subset $X \subseteq V(G)$ we define $N(X) = \cup_{v \in X} N(v) - X$. A set of vertices $A \subseteq V(G)$ is called an independent set iff every vertex of it is not adjacent to any other. We use $\alpha$to denote the cardinality of a maximum independent set of vertices of G. A cycle C of G is called a dominating cycle, or briefly D-cycle, if $V(G) - V(C)$ is an independent set of vertices in G. We denote $NC(G) = \min\{|N(u) \cup N(v)|: uv \notin E(G)\}$and $NC2(G) = \min\{|N(u) \cup N(v)|: d(u,v) = 2\}$, where $d(u,v)$ is length of the shortest path joining u and v if G is not complete. We set $NC(G) = NC2(G) = n - 1$ if G is complete. Moreover, we define $NC_k(G) = \min\{|\cup_{i=1}^{k} N(v_i)|\}$ and $\sigma_k(G) = \min\{d(v_1) + \cdots + d(v_k)\}$ where $\{v_1, \ldots, v_k\}$ is an independent set of vertices in case there exists the independent set $\{v_1, \ldots, v_k\}$, otherwise $NC_k(G) = n - \alpha$ and $\sigma_k(G) = k(n - \alpha)$. Sometimes we write $N(a,b)$ instead of $N(a) \cup N(b)$.

A graph G is called to be 1-tough if for every nonempty set$S \subseteq V(G)$, the number component of $G - S$, denoted by$\omega(G - S)$, less than or equal to$|S|$. Two disjoint paths of$G$ is said to be to *adjacent* if there is at least one vertex of a path adjacent to an vertex of the other.Denote by $c(G)$ the circumference (the length of a longest cycle) of G, the following bound on$c(G)$ due to Bauer, Fan and Veldman

Theorem 1 (Theorem 26 in [1]).*If $G$is 1-tough and$\sigma_3 \geq n$, then$c(G) \geq \min\{n, 2NC2(G)\}$.*

Vu Dinh Hoa strengthened this result in [4].

Theorem 2 (Theorem 2 in [4]).*IfG is a 1-tough graph with$\sigma_3 \geq n \geq 3$, then$c(G) \geq \min\{n, 2NC_{\sigma_3 - n+5} + 2\}$.*

Bauer, Veldman and Fan also conjectured :

Conjecture 1 (Conjecture 27 in [1]).*If $G$is 1-tough and$\sigma_3 \geq n$, then$c(G) \geq \min\{n, 2NC2(G) + 4\}$.*

The conjecture remains unsolved even some people have tried to prove it (see [9] ...). The proof of Tri Lai [9] is not correct, for example the proof for the **Proposition 26**, and the definition of **bad path** is not strong enough. In the mean time, a lot of new bounds for the length of longest cycle was found. For examples:

Theorem 3 (Theorem 3 in [5]).*Let$G$ be a graph of order$n$. If $G$is 3-connected then$c(G) \geq \min\{n, \frac{3(NC+1)}{2}\}$; if$G$ is 4-connected, then$c(G) \geq \min\{n, 2NC\}$. Furthermore, these bounds are both sharp.*

Theorem 4 (Theorem 2 in [8]).*If G is 3-connected$K_{1,3}$-free graph of order$n$ such that$NC2 \geq (2n-6)/3$ then$G$ is hamiltonian.*

Theorem 5 (Theorem 4 in [6]) *Let$G$ be a 2-connected graph such that$\max\{d(u), d(v)\} \geq c/2$ for each pair of nonadjacent vertices$u$ and$v$ in an induded claw, and$|N(x) \cap N(y)| \geq 2$ for each pair of nonadjacent vertices$x$ and $y$in an induced modified claw. Then$G$ contains either Hamiltonian cycle or a cycle of length at least$c$.*

Our goal in this paper is to prove the conjecture posed by Bauer, Veldman and Fan in [1].

II. PRELIMINARIES

For what follows, we assume that G is1-tough with $\sigma_3 \geq n$ and nonhamiltonian. For a longest cycle C, we denote by $\mu(C) = \max\{d(v): v \notin C\}$ and $\mu(G) = \max\{\mu(C): l(C) = c(G)\}$ where $l(C)$ is the length of C. We consider a longest cycle C and a vertex $v \notin C$ such that $\mu(C) = \mu(G)$ and $d(v) = \mu(C)$. On $\overrightarrow{C}$ (C with a given orientation), we denote the predecessor and successor of $x \in C$ (along the direction of C) by x^-, x^+ and $x^{++} = (x^+)^+$, similarly, $x^{--} = (x^-)^-$. In general, $x^{+i} = (x^{+(i-1)})^+$ and $x^{-i} = (x^{-(i-1)})^-$. Let $S = N(v)^+ \cap N(v)^-$. The arc joining two vertices x and y of C, along the direction of $\overrightarrow{C}$, is denoted by $x\overrightarrow{C}y$. Similarly, the arc joining two vertices x and y of C, along the direction of $\overleftarrow{C}$, is denoted by $x\overleftarrow{C}y$. Moreover, for any $A \subseteq V(G)$, we write $A^+ = \{x^+: x \in A\}$ and $A^- = \{x^-: x \in A\}$.We begin with following lemmas:

Lemma 1 (Theorem 5 in [2]). *Every longest cycle C is a D-cycle.*

Lemma 2 (Proof of Theorem 9 in [2]). $d(v) \geq \frac{\sigma_3}{3}$.

Lemma 3 (Lemma 9 in [4]). $|S| \geq \sigma_3 - n + 4$.

For what follows we use a lemma of Woodall, sometimes called Hopping Lemma, in [7]:

Lemma 4. *Let $\overrightarrow{C}$ be a cycle of length m in a graph G. Assume that G contains no cycle of length $m+1$ and no cycle C^* of length m with $\omega(G - C^*) < \omega(G - C)$ and v is an isolated vertices of $G - C$. Set $Y_0 = \emptyset$ and for $i > 1$:*

$$X_{i+1} = N(Y_i \cup \{v\}),$$
$$Y_{i+1} = (X_i \cap V(C))^+ \cap (X_i \cap V(C))^-.$$

Set $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=1}^{\infty} Y_i$. Then

(a) $X \subseteq V(C)$,

(b) If $x_1, x_2 \in X$, then $x_1^+ \neq x_2$.

(c) $X \cap Y = \emptyset$.

By (a), (b) and (c) of Lemma 4, respectively, we easily conclude that:

Corollary 1. *Assume that G, C, X, Y satisfy the hypothesis of Lemma 4, then:*

(a) There are no edges joining a vertex of Y with a vertex of $G - C$.

(b) $X \cap X^+ = \emptyset$, and therefore $l(C) \geq 2|X|$.

(c) Y is an independent set of vertices.

Assume that X, Y are the vertex sets mentioned in the assumption of Lemma 4. First, we fixed a vertex $s_0 \in S$ and denote the vertices of X by $x_1, x_2, \ldots, x_m (m = |X|)$, occurring consecutively on $\overrightarrow{C}$ such that $x_1 = s_0^+$ and $x_m = s_0^-$. For $1 \leq i \leq m-1$ we write $u_i = x_i^+$ and $w_i = x_{i+1}^-$. By Lemma 4, $x_i^+ \neq x_{i+1}$ and therefore $C - X$ is in fact the union of $|X|$ disjoint paths $u_i\overrightarrow{C}w_i$ ($i = \overline{1,m}$) on C, calling *arcs*. An arc with p vertices is called a *p-arc*. $u_i\overrightarrow{C}w_i$ is an 1-arc if and only if $u_i \equiv w_i \in Y$. The next lemma can be proved using Corollary 1 and Lemma 1.

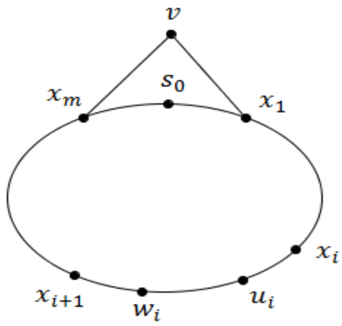

Fig.1 Illustrating the vertices on cycle C.

Lemma 5.

(a) $Y \cup V(G - C)$ is an independent set of vertices.

(b) $l(C) \geq 2|X| + 2$.

Proof.

(a) By (a) of Corollary 1, there is no edge joining a vertex of Y with a vertex of $V(G - C)$. By (c) of Corollary 1, Y is an independent set of vertices. By Lemma 1, $V(G - C)$ is an independent set of vertices. Thus, $Y \cup V(G - C)$ is an independent set of vertices.

(b) By (b) of Corollary 1, $l(C) \geq 2|X|$. If $l(C) \leq 2|X| + 1$ then $C - X$ is an union of $|X|$ 1-arcs or an union of one 2-arc and $(|X| - 1)$ 1-arcs. The 1-arcs of $C - X$ are in fact the vertices of Y. By (a), the graph $G - X$ contains at least $|X| + 1$ components, a contradiction to the toughness of G.

By the toughness of G, by $X = N(Y)$ and by (a) of Lemma 5, we easily conclude that:

Corollary 2. *$C - X$ contains at least two arcs with length ≥ 2.*

By $d(v, s) = 2$ for any $s \in S$ and by $N(s) \cup N(v) \subseteq X$, $NC2 \leq |N(s) \cup N(v)| \leq |X|$. If $NC2 \neq |X|$ then by Lemma 5, $c(G) \geq 2|X| + 2 \geq 2NC2 + 4$ and Conjecture 1 is proved. For what follows we assume that $NC2 = |X|$ and $c(G) \leq 2|X| + 3$ and will show that it leads to a contradiction. By $NC2 = |X|$, $X = N(s) \cup N(v)$ for any $s \in S$ and we conclude from Lemma 4 in [4] that:

Lemma 6. *$V(G - C) \cup X^+$ is an independent set. Similarly, $V(G - C) \cup X^-$ is an independent set.*

The following definition will help us to get shorter proofs.

Definition 1. *A path P in G is called a bad-path if it has one of the following two forms:*

1. P consists of all vertices in $x_1\overrightarrow{C}w_{j-1}$, and the ends of P are $x_k \in N(v) \cap N(s_0)$ and $x_i \in X$ for $1 \le k, i < j \le m$ and $i \ne k$.

2. P consists of all vertices in $x_m\overleftarrow{C}u_{i+1}$, and the ends of P are $x_k \in N(v) \cap N(s_0)$ and $x_j \in X$ for $1 \le i < j, k \le m$ and $j \ne k$.

Lemma 7. *There are no bad-paths in G.*

Proof. Assume the contrary that G has a bad-path P. Without loss of generality, we assume that P is a bad-path of form 1. We will show a cycle C' is longer than C, then get a contradiction. As remark before that $X = N(s) \cup N(v)$ for any $s \in S$ and specially $X = N(s_0) \cup N(v)$. Therefore, there are only four possible cases as follows. In each case we will get a contradiction by constructing a cycle C' which is longer than C.

Case 1: $x_i, x_j \in N(s_0)$, then $C' = (x_k P x_i s_0 x_j \overrightarrow{C} x_m v x_k)$.

Case 2: $x_i, x_j \in N(v)$, then $C' = (x_k P x_i v x_j \overrightarrow{C} s_0 x_k)$.

Case 3: $x_i \in N(s_0), x_j \in N(v)$, then $C' = (x_k P x_i s_0 \overleftarrow{C} x_j v x_k)$.

Case 4: $x_i \in N(v), x_j \in N(s_0)$, then $C' = (x_k P x_i v x_m \overleftarrow{C} x_j s_0 x_k)$.

Lemma 8. *If $w_i u_j \in E(G)$ for $1 \le i < j < m$ then $x_{i+1}, x_j \in N(v) - N(s_0)$ or $x_{i+1}, x_j \in N(s_0) - N(v)$.*

Proof. Assume to the contrary that $x_{i+1} \in N(s_0), x_j \in N(v)$ or $x_{i+1} \in N(v), x_j \in N(s_0)$ then we have a cycle $C' = (v x_j \overleftarrow{C} x_{i+1} s_0 \overrightarrow{C} w_i u_j \overrightarrow{C} x_m v)$ or $C' = (v x_1 \overrightarrow{C} w_i u_j \overrightarrow{C} s_0 x_j \overleftarrow{C} x_{i+1} v)$ respectively, is longer than C, a contradiction.

Lemma 9. *If $w_i u_j \in E(G)$ for $1 \le i < j < m$ then $x_{i+1}, x_j \notin N(u_1) \cup N(w_{m-1})$.*

Proof. If $x_{i+1} \in N(u_1)$ then $P = (x_m \overleftarrow{C} u_j w_i \overleftarrow{C} u_1 x_{i+1} \overrightarrow{C} x_j)$ is a bad-path of form 2, and if $x_j \in N(u_1)$ then $P = (x_m \overleftarrow{C} u_j w_i \overleftarrow{C} u_1 x_j \overleftarrow{C} x_{i+1})$ is also a bad-path of form 2, which contradicts to Lemma 7. Thus, we have $x_{i+1}, x_j \notin N(u_1)$. Similarly, by reversing the direction of C, we have $x_{i+1}, x_j \notin N(w_{m-1})$.

Lemma 10. *Assume that $u_i w_j \in E(G)$ for some $1 \le i < j < m$. Then $x_i, x_{j+1} \notin N(u_k)$ for every $i < k \le j$. Analogously, $x_i, x_{j+1} \notin N(w_k)$ for every $i \le k < j$.*

Proof. If $x_i u_k \in E(G)$ for some $i < k \le j$ then $P = (x_1 \overrightarrow{C} x_i u_k \overrightarrow{C} w_j u_i \overrightarrow{C} x_k)$ is a bad-path, which contradicts to Lemma 7. If $x_{j+1} u_k \in E(G)$ for some $i < k \le j$ then $P = (x_m \overleftarrow{C} x_{j+1} u_k \overrightarrow{C} w_j u_i \overrightarrow{C} x_k)$ is a bad-path, a contradiction. Thus, we have $x_i, x_{j+1} \notin N(u_k)$ for every $i < k \le j$. Similarly, by reversing the direction of C, we have $x_i, x_{j+1} \notin N(w_k)$ for every $i \le k < j$.

Lemma 11. *Assume that $u_i w_j \in E(G)$ for some $1 \le i < j < m$. If there exists a vertex $z \in u_i^+ \overrightarrow{C} w_j^-$ such that $z u_1 \in E(G)$ and $i > 1$ then $z^+, z^- \notin N(x_1)$. Analogously, if there exists a vertex $z \in u_i^+ \overrightarrow{C} w_j^-$ such that $z w_{m-1} \in E(G)$ and $j < m-1$ then $z^+, z^- \notin N(x_m)$.*

Proof. Assume that z is vertex in $u_i^+ \overrightarrow{C} w_j^-$ such that $z u_1 \in E(G)$ and $i > 1$. If $z^- x_1 \in E(G)$ then we have $P = \left(x_1 z^- \overleftarrow{C} u_i w_j \overleftarrow{C} z u_1 \overrightarrow{C} x_i\right)$ is a bad-path, which contradicts to Lemma 7. If $z^+ x_1 \in E(G)$ then $P = (x_1 z^+ \overrightarrow{C} w_j u_i \overrightarrow{C} z u_1 \overrightarrow{C} x_i)$ is a bad-path, a contradiction. Thus, we have $z^+, z^- \notin N(x_1)$. Similarly, if $z w_{m-1} \in E(G)$ and $j < m-1$ then $z^+, z^- \notin N(x_m)$.

Lemma 12. *Assume that $u_i w_j \in E(G)$ for some $1 \le i < j < m$. If there exists a vertex $x_k \in u_i \overrightarrow{C} w_j$ such that $x_k \in N(v) \cap N(s_0)$ then $x_1, x_m \notin N(w_{k-1}) \cup N(u_k)$.*

Proof. Let x_k be a vertex in $N(v) \cap N(s_0) \cap u_i \overrightarrow{C} w_j$. If $x_1 w_{k-1} \in E(G)$ then $P = (x_k \overrightarrow{C} w_j u_i \overrightarrow{C} w_{k-1} x_1 \overrightarrow{C} x_i)$ is a bad-path, which contradicts to Lemma 7. If $x_m w_{k-1} \in E(G)$ then $P = (x_k \overrightarrow{C} w_j u_i \overrightarrow{C} w_{k-1} x_m \overleftarrow{C} x_{j+1})$ is a bad-path, a contradiction. Thus, $x_1, x_m \notin N(w_{k-1})$. Similarly, we have $x_1, x_m \notin N(u_k)$.

Lemma 13. *Assume that C has a p-arc $u_i \overrightarrow{C} w_i$ and a q-arc $u_j \overrightarrow{C} w_j$ with $min(p, q) = 2$, $i \ne j$. Then $\{u_i w_j, u_j w_i\} \not\subseteq E(G)$.*

Proof. Assume to the contrary that $u_i w_j, u_j w_i \in E(G)$. Without loss of generality, we assume that $p = 2$, then we have $P = (x_1 \overrightarrow{C} u_i w_j \overleftarrow{C} u_j w_i \overrightarrow{C} x_j)$ is a bad-path, which contradicts to Lemma 7.

Lemma 14. *Assume that C has a p-arc $u_i \overrightarrow{C} w_i$ and a q-arc $u_j \overrightarrow{C} w_j$ with $p, q \ge 2, i < j$, such that $u_i w_j \in E(G)$ and there are no k-arcs ($k \ge 2$) on $x_{i+1} \overrightarrow{C} x_j$. If $N(w_i) \subseteq X \cup \{w_i^-\}$ or $N(u_j) \subseteq X \cup \{u_j^+\}$, then $x_{i+1} \equiv x_j$.*

Proof. Assume to the contrary that $x_{i+1} \ne x_j$. Without loss of generality, we assume that $N(w_i) \subseteq X \cup \{w_i^-\}$. Since there are no k-arcs ($k \ge 2$) on $x_{i+1} \overrightarrow{C} x_j$, $u_{i+1} \in Y$. By Lemma 6, $d(w_i, u_{i+1}) = 2$. By Lemma 10 for $k = i+1$, we have $x_i, x_{j+1} \notin N(w_i, u_{i+1})$. Thus, by Lemma 6 we have $N(w_i, u_{i+1}) \subseteq X \cup \{w_i^-\} - \{x_i, x_{j+1}\}$. Therefore, $|X| - 1 \ge |N(w_i, u_{i+1})| \ge NC2(G) = |X|$, a contradiction.

Lemma 15. *Assume that C has p-arc $u_i \overrightarrow{C} w_i$ and a q-arc $u_j \overrightarrow{C} w_j$ with $p \ge 2, q \ge 2, i < j$ such that there are no k-arcs ($k \ge 2$) on $x_1 \overrightarrow{C} x_i \cup x_{i+1} \overrightarrow{C} x_m$. If $N(u_i) \subseteq X \cup \{u_i^+\}$ or $N\left(w_j\right) \subseteq X \cup \{w_j^-\}$ then $w_i u_j \notin E(G)$.*

Proof. Assume to the contrary that $w_i u_j \in E(G)$. Without loss of generality, we assume that $N\left(w_j\right) \subseteq X \cup \{w_j^-\}$. By Lemma 8, we have $x_{i+1}, x_j \in N(v) - N(s_0)$ or $x_{i+1}, x_j \in N(s_0) - N(v)$. Let y be a vertex in $\{v, s_0\}$ such that $x_{i+1}, x_j \notin$

$N(y)$. Clearly, $d(y,u_1)=d(y,w_{m-1})=2$, so $|N(y,u_1)|, |N(y,w_{m-1})| \geq NC2(G)=|X|$. If $x_i \neq x_1$, then $u_1 \in Y$. By Lemma 9, we have $x_{i+1}, x_j \notin N(u_1)$, and therefore $N(y,u_1) \subseteq X-\{x_{i+1},x_j\}$, a contradiction. Thus, $x_i \equiv x_1$. Similarly, we have $x_{j+1} \equiv x_m$ and $w_j \equiv w_{m-1}$. By Lemma 9, we have $x_{i+1}, x_j \notin N(w_{m-1})$, so $N(y,w_{m-1}) \subseteq X \cup \{w_{m-1}^-\}-\{x_{i+1},x_j\}$. If $x_{i+1} \neq x_j$, then $|N(y,w_{m-1})| \leq |X \cup \{w_{m-1}^-\}-\{x_{i+1},x_j\}| \leq |X|-1$, a contradiction. Thus $x_{i+1} \equiv x_j$, and therefore $|S|=1$, which contradicts to Lemma 3 that $|S| \geq \sigma_3-n+4 \geq 4$.

By $c(G) \leq 2|X|+3$ and by (b) of Lemma 5, either $c(G)=2|X|+3$ or $c(G)=2|X|+2$. We consider two cases:

Case 1. $c(G)=2|X|+2$. In this case, by Corollary 2, $C-X$ contains two 2-arc and some 1-arcs.

Case 2. $c(G)=2|X|+3$. In this case, by Corollary 2, there are only two possible cases:

Case 2.1. $C-X$ contains one 2-arc, one 3-arc and some 1-arcs.

Case 2.2. $C-X$ contains three 2-arc and some 1-arcs.

III. PROOF FOR THE CASES

A. Case 1. $c(G)=2|X|+2$, $C-X$ contains two 2-arc and some 1-arcs.

Assume that the two 2-arcs are u_iw_i and u_jw_j with $1 \leq i < j < m$. We have the following Claims (from Claim 1 to Claim 4):

Claim 1. $u_iw_j \in E(G)$ and $u_jw_i \notin E(G)$.

Proof. Assume that $u_iw_j \notin E(G)$, then $u_jw_i \in E(G)$, otherwise, by Lemma 6, $\omega(G-X) \geq |X|+1$, which contradicts the toughness of G. By Lemma 6, $N(u_i) \subseteq X \cup \{w_i\} = X \cup \{u_i^+\}$, and by Lemma 15, $u_jw_i \notin E(G)$, a contradiction.

Thus, $u_iw_j \in E(G)$, and by Lemma 13, $u_jw_i \notin E(G)$.

By Lemma 6, Lemma 14 and Claim 1, we get:

Claim 2. $x_{i+1} \equiv x_j$.

Note: by Claim 1 and Claim 2, $d(w_i,u_j)=2$. By Lemma 10, we have $x_i, x_{j+1} \notin N(w_i,u_j)$.

Claim 3. $x_j \notin N(v) \cap N(s_0)$.

Proof. Assume the contrary that $x_j \in N(v) \cap N(s_0)$. By Lemma 12, $x_1, x_m \notin N(w_i,u_j)$. By Lemma 6, $N(w_i,u_j) \subseteq X \cup \{u_i,w_j\}-\{x_1,x_m,x_i,x_{j+1}\}$. By $|N(w_i,u_j)|=|N(w_i) \cup N(u_j)| \geq NC2(G)=|X|$, we have $x_i \equiv x_1$ and $x_{j+1} \equiv x_m$. Thus, $|S|=1$, which contradicts to Lemma 3 that $|S| \geq \sigma_3-n+4 \geq 4$.

Claim 4. $x_i \equiv x_1$ and $x_{j+1} \equiv x_m$.

Proof. Assume that $x_i \neq x_1$, then $u_1 \in Y$. By Claim 3, there exists $y \in \{v,s_0\}$ such that $x_jy \notin E(G)$. Clearly, $d(u_1,y)=2$. If $u_1x_j \notin E(G)$, then $N(u_1,y) \subseteq X-\{x_j\}$, and therefore $|N(u_1,y)| \leq |X|-1$, which contradicts $|N(u_1,y)| \geq NC2(G)=|X|$. Thus, $u_1x_j \in E(G)$. By Lemma 11, $x_1 \notin N(w_i,u_j)$, and by Lemma 6, $N(w_i,u_j) \subseteq X \cup \{u_i,w_j\}-\{x_1,x_i,x_{j+1}\}$. Therefore, $|N(w_i,u_j)|=|X|-1$, which contradicts $d(w_i,u_j)=2$ as noted above.

Thus $x_i \equiv x_1$. Similarly, we have $x_{j+1} \equiv x_m$.

By Claim 2 and Claim 4, $|S|=1$, which contradicts Lemma 3. So, the Case 1 does not happen.

B. Case 2. $c(G)=2|X|+3$. In this case, there are only two possible cases:

Case 2.1. $C-X$ contains one 2-arc, one 3-arc and some 1-arcs.

Assume that the 3-arc is $u_iu_i^+w_i$ and the 2-arc is u_jw_j. Without loss of generality, we assume that $i<j$. We have two following Propositions:

Proposition 1. $u_iw_i \notin E(G)$.

Proof. Assume to the contrary that $u_iw_i \in E(G)$. We have the following Claims (from Claim 5 to Claim 10):

Claim 5. $u_i^+u_j, u_i^+w_j \notin E(G)$.

Proof. If $u_i^+u_j \in E(G)$ then $P=(x_m\overleftarrow{C}u_ju_i^+u_iw_i\overrightarrow{C}x_j)$ is a bad-path, which contradicts Lemma 7. If $u_i^+w_j \in E(G)$ then $P=(x_1\overrightarrow{C}u_iw_iu_i^+w_j\overleftarrow{C}x_{i+1})$ is a bad-path, which contradicts Lemma 7.

Claim 6. $u_iw_j \in E(G)$ and $u_jw_i \notin E(G)$. *Moreover,* $x_{i+1} \equiv x_j$.

Proof. If $u_iw_j, u_jw_i \notin E(G)$, then by Lemma 6 and by Claim 5, $\omega(G-X) \geq |X|+1$, so G is not 1-tough, a contradiction.

If $u_jw_i \in E(G)$, then by Lemma 13, $u_iw_j \notin E(G)$. Therefore, by Lemma 6 and by Claim 5, $N(w_j) \subseteq X \cup \{u_j\}$. By Lemma 15 we have $u_jw_i \notin E(G)$, a contradiction.

Thus, $u_jw_i \notin E(G)$. By the toughness of G, by Claim 5 and by Lemma 6, $u_iw_j \in E(G)$. By $u_jw_i \notin E(G)$ and by Lemma 6, $N(u_j) \subseteq X \cup \{w_j\}$. By Lemma 14, $x_{i+1} \equiv x_j$.

Claim 7. $d(w_i,u_j)=2$ *and* $x_i, x_{j+1} \notin N(w_i,u_j)$.

Proof. By Claim 6, $d(w_i,u_j)=2$. By Lemma 10, $x_i, x_{j+1} \notin N(w_i,u_j)$.

Claim 8. $x_j \notin N(v) \cap N(s_0)$.

Proof. Assume to the contrary that $x_j \in N(v) \cap N(s_0)$. Since Lemma 12 we have $x_1, x_m \notin N(w_i,u_j)$. By Lemma 6 and by Claim 7, $N(w_i,u_j) \subseteq X \cup \{u_i,u_i^+,w_j\}-\{x_1,x_m,x_i,x_{j+1}\}$. Because $|N(w_i,u_j)| \geq NC2(G)=|X|$, so $x_i \equiv x_1$ or $x_{j+1} \equiv x_m$.

If $x_i \equiv x_1$, then $x_{j+1} \neq x_m$ because of $|S| \geq 4$, and there are at least three 1-arcs on $x_{j+1}\overrightarrow{C}x_m$. Since $N(w_i,u_j) \subseteq$

$X \cup \{u_i, u_i^+, w_j\} - \{x_1, x_m, x_{j+1}\}$, so every vertex $x_k (j+1 < k < m) \in N(w_i, u_j)$, and we observe that $u_k \notin N(x_{j+1})$. Indeed, suppose otherwise that $u_k x_{j+1} \in E(G)$, by Claim 6 we have:

(1) If $x_k w_i \in E(G)$ then $P = (x_m \overleftarrow{C} u_k x_{j+1} \overrightarrow{C} x_k w_i u_i^+ u_i w_j u_j x_j)$ is a bad-path, a contradiction.

(2) If $x_k u_j \in E(G)$ then $P = (x_m \overleftarrow{C} u_k x_{j+1} \overrightarrow{C} x_k u_j w_j u_i \overrightarrow{C} x_j)$ is a bad-path, a contradiction.

Therefore, $u_k \notin N(x_{j+1})$. Clearly that $u_k \in Y$ for $j+1 < k < m$, so $d(u_{j+2}, u_{j+3}) = 2$ and $N(u_{j+2}, u_{j+3}) \subseteq X - \{x_{j+1}\}$, this implies that $NC2(G) \leq |X| - 1$, a contradiction.

Similarly, if $x_{j+1} \equiv x_m$, then we get a contradiction.

Note: By Claim 8, let $y \in \{v, s_0\}$ such that $x_j \notin N(y)$.

Claim 9. $x_1, x_m \notin N(w_i, u_j)$.

Proof. If $x_i \equiv x_1$ then by Claim 7, it is clear that $x_1 \notin N(w_i, u_j)$. We consider the case $x_i \neq x_1$.

Assume that $u_1 x_j \notin E(G)$. Clearly that $d(y, u_1) = 2$ and $u_1 \in Y$, so $N(y, u_1) \subseteq X - \{x_j\}$, this implies that $|X| - 1 \geq N(y, u_1) \geq NC2(G) = |X|$, a contradiction. Therefore, $u_1 x_j \in E(G)$. By Lemma 11, we have $x_1 \notin N(w_i, u_j)$. Similarly, we have $x_m \notin N(w_i, u_j)$.

Claim 10. $x_i \equiv x_1$ or $x_{j+1} \equiv x_m$.

Proof. Assume to the contrary that $x_i \neq x_1$ and $x_{j+1} \neq x_m$. By Claim 7 and Claim 9, $x_i, x_{j+1}, x_1, x_m \notin N(w_i, u_j)$. By Lemma 6, Claim 5 and Claim 6, $N(w_i, u_j) \subseteq X \cup \{u_i, u_i^+, w_j\} - \{x_1, x_m, x_i, x_{j+1}\}$. Thus, $|X| - 1 \geq N(w_i, u_j) \geq NC2(G) = |X|$, a contradiction.

If $x_i \equiv x_1$, then by $|S| \geq \sigma_3 - n + 4 \geq 4$ and by $x_{i+1} \equiv x_j$ (see Claim 6), we have $x_{j+1} \neq x_m$ and there are at least three 1-arcs on $x_{j+1} \overrightarrow{C} x_m$. Applying similar arguments of the proof for $u_k \in x_{j+1} \overrightarrow{C} x_m$ of Claim 8, we have $u_k \in Y$ and $u_k \notin N(x_{j+1})$ for every $j+1 < k < m$, so $d(u_{j+2}, u_{j+3}) = 2$ and $N(u_{j+2}, u_{j+3}) \subseteq X - \{x_{j+1}\}$, this implies that $NC2(G) \leq |X| - 1$, a contradiction. Thus $x_i \neq x_1$. Similarly, $x_{j+1} \neq x_m$, which contradicts Claim 10.

This finishes the proof of the Proposition 1.

Proposition 2. $u_j w_i \notin E(G)$ and $u_i w_j \in E(G)$.

Proof. If $u_j w_i \in E(G)$, by Lemma 13, $u_i w_j \notin E(G)$. Therefore, by Lemma 6 and by Proposition 1, $N(u_i) \subseteq X \cup \{u_i^+\}$. By Lemma 15, $u_j w_i \notin E(G)$, a contradiction. Thus, $u_j w_i \notin E(G)$.

If $u_i w_j \notin E(G)$, then by Lemma 6 and by Proposition 1, $\omega(G - (X \cup \{u_i^+\})) \geq |X| + 2$, which contradicts the toughness of G. Thus, $u_i w_j \in E(G)$.

By Lemma 6 and by Proposition 1, $N(w_i) \subseteq X \cup \{u_i^+\} = X \cup \{w_i^-\}$. . By Lemma 14, $x_{i+1} \equiv x_j$, and by Proposition 2, $d(w_i, u_j) = 2$. By Lemma 10, $x_i, x_{j+1} \notin N(w_i, u_j)$.

By using similar arguments in the proofs of Claim 8 and of Claim 9, $x_j \notin N(v) \cap N(s_0)$, and $x_1, x_m \notin N(w_i, u_j)$. Therefore, $N(w_i, u_j) \subseteq X \cup \{u_i^+, w_j\} - \{x_1, x_m, x_i, x_{j+1}\}$. By $NC2(G) = |X|$, we have $x_i \equiv x_1$ and $x_{j+1} \equiv x_m$. This implies that $|S| = 1$, which contradicts Lemma 3 that $|S| \geq \sigma_3 - n + 4 \geq 4$.

Thus, the Case 2.1. does not happen.

Case 2.2. $C - X$ contains three 2-arcs and some 1-arcs on C.

Assume that the three 2-arcs are $u_i w_i, u_j w_j, u_k w_k$ with $i < j < k$. We will prove that the three 2-arcs are not adjacent. We have two following Propositions:

Proposition 3. $u_i w_i$ *is not adjacent to* $u_k w_k$.

Proof. Assume the contrary that $u_i w_i$ is adjacent to $u_k w_k$. We have the following Claims (from Claim 11 to Claim 15):

Claim 11. $u_i w_k \in E(G)$ *and* $w_i u_k \notin E(G)$.

Proof. Assume otherwise that $u_i w_k \notin E(G)$, then $w_i u_k \in E(G)$ since $u_i w_i$ is adjacent to $u_k w_k$.

If $w_k u_j \notin E(G)$, then by Lemma 6, $N(w_k) \subseteq X \cup \{u_k\}$. By Lemma 15, applying for $u_i \overrightarrow{C} w_i$ and $u_k \overrightarrow{C} w_k$ we have $w_i u_k \notin E(G)$, which contradicts to the fact that $w_i u_k \in E(G)$. Thus, $w_k u_j \in E(G)$. Similarly, we have $u_i w_j \in E(G)$.

By Lemma 9 applying for $w_i u_k \in E(G)$, $x_{i+1}, x_k \notin N(u_1, w_{m-1})$. By Lemma 8 we have $x_{i+1}, x_k \in N(v) - N(s_0)$ or $x_{i+1}, x_k \in N(s_0) - N(v)$. Let $y \in \{v, s_0\}$ such that $x_{i+1}, x_k \notin N(y)$.

If $x_i \neq x_1$ then $u_1 \in Y$ and $N(y, u_1) \subseteq X - \{x_{i+1}, x_k\}$, so $|N(y, u_1)| \leq |X| - 2$. However, since $d(y, u_1) = 2$, we get $|N(y, u_1)| \geq NC2(G) = |X|$, a contradiction. Thus, $x_i \equiv x_1$. Similarly, we have $x_{k+1} \equiv x_m$.

Because $|S| \geq 4$, $x_{i+1} \overrightarrow{C} x_j$ or $x_{j+1} \overrightarrow{C} x_k$ has at least two 1-arcs. Without loss of generality, we assume that $x_{i+1} \overrightarrow{C} x_j$ has at least two 1-arcs, so $u_{i+1}, u_{i+2} \in Y$. By $u_i w_j \in E(G)$ and by Lemma 10, $x_1, x_{j+1} \notin N(u_{i+1}, u_{i+2})$. Therefore, $N(u_{i+1}, u_{i+2}) \subseteq X - \{x_1, x_{j+1}\}$, so $|N(u_{i+1}, u_{i+2})| \leq |X| - 2$. Because $d(u_{i+1}, u_{i+2}) = 2$, we have $|N(u_{i+1}, u_{i+2})| \geq NC2(G) = |X|$, a contradiction.

Thus, $u_i w_k \in E(G)$. By Lemma 13, we have $w_i u_k \notin E(G)$.

Claim 12. $u_k w_j \notin E(G)$ *and* $w_i u_j \notin E(G)$.

Proof. If $u_k w_j \in E(G)$ then $P = (x_m \overleftarrow{C} w_k u_i \overrightarrow{C} w_j u_k \overleftarrow{C} x_{j+1})$ is a bad-path, a contradiction.

If $w_i u_j \in E(G)$ then $P = (x_1 \overrightarrow{C} u_i w_k \overleftarrow{C} u_j w_i \overrightarrow{C} x_j)$ is a bad-path, a contradiction.

Claim 13. $x_{i+1} \equiv x_j$ *and* $x_{j+1} \equiv x_k$.

Proof. Assume that $x_{i+1} \neq x_j$. Then $u_{i+1} \in Y$ and $d(w_i, u_{i+1}) = 2$. By Lemma 10, $x_i, x_{k+1} \notin N(w_i, u_{i+1})$. By Lemma 6 and by Claims 11, 12, we have $N(w_i, u_{i+1}) \subseteq X \cup \{u_i\} - \{x_i, x_{k+1}\}$, therefore $|X| - 1 \geq |N(w_i, u_{i+1})| \geq NC2(G) = |X|$, a contradiction.

Thus $x_{i+1} \equiv x_j$. Similarly, we have $x_{j+1} \equiv x_k$.

Note: Now we consider the pair $\{w_i, u_j\}$ with distance 2. By Lemma 10, we have $x_i, x_{k+1} \notin N(w_i, u_j)$.

Claim 14.$x_1, x_m \notin N(w_i, u_j)$.

Proof. First, we prove that $x_1 \notin N(w_i, u_j)$. The proof for the case $x_m \notin N(w_i, u_j)$ is similar. If $x_i \equiv x_1$ then by Lemma 10, $x_1 \notin N(w_i, u_j)$. For what follows, we assume that $x_i \neq x_1$. Clearly, $u_1 \in Y$.

If $x_j \in N(v) \cap N(s_0)$, then by Lemma 12, $x_1 \notin N(w_i, u_j)$.

If $x_j \notin N(v) \cap N(s_0)$. Let $y \in \{s_0, v\}$ such that $x_j \notin N(y)$. Clearly, $d(y, u_1) = 2$. If $u_1 x_j \notin E(G)$, then $N(y, u_1) \subseteq X - \{x_j\}$ and therefore $|N(y, u_1)| \leq |X| - 1$, which contradicts to the fact that $|N(y, u_1)| \geq NC2(G) = |X|$. Thus, $u_1 x_j \in E(G)$. By Lemma 11, we have $x_1 \notin N(w_i, u_j)$.

Claim 15.$x_i \equiv x_1$ or $x_{k+1} \equiv x_m$.

Proof. Assume to the contrary that $x_i \neq x_1$ and $x_{k+1} \neq x_m$. By Lemmas 6, 10 and by Claims 12, 14, we have $N(w_i, u_j) \subseteq X \cup \{u_i, w_j, w_k\} - \{x_1, x_m, x_i, x_{k+1}\}$. By Claim 13, $d(w_i, u_j) = 2$, we get $|X| - 1 \geq |N(w_i, u_j)| \geq NC2(G) = |X|$, a contradiction.

Assume without loss of generality that $x_i \equiv x_1$. By Claim 13 and by $|S| \geq 4$, $x_{k+1} \neq x_m$ and there are at least three 1-arcs on $x_{k+1}\overrightarrow{C}x_m$. Since $|X| = NC2(G) \leq |N(w_i, u_j)| \leq |X \cup \{u_i, w_j, w_k\} - \{x_1, x_m, x_{k+1}\}| = |X|$, so $x_t \in N(w_i, u_j)$ for every $k + 1 < t < m$, and we will show that $u_t x_{k+1} \notin E(G)$. Indeed, assume otherwise that $u_t x_{k+1} \in E(G)$. We have two cases:

(1) If $x_t w_i \in E(G)$ then $P = (x_m\overleftarrow{C}u_t x_{k+1}\overrightarrow{C}x_t w_i u_i w_k\overleftarrow{C}x_j)$ is a bad-path, a contradiction.

(2) If $x_t u_j \in E(G)$ then $P = (x_m\overleftarrow{C}u_t x_{k+1}\overrightarrow{C}x_t u_j\overrightarrow{C}w_k u_i w_i x_j)$ is a bad-path, a contradiction.

Clearly, $u_{k+2}, u_{k+3} \in Y$ and $N(u_{k+2}, u_{k+3}) \subseteq X - \{x_{k+1}\}$ and $d(u_{k+2}, u_{k+3}) = 2$, this implies that $|X| = NC2(G) \leq |N(u_{k+2}, u_{k+3})| \leq |X - \{x_{k+1}\}| = |X| - 1$, a contradiction.

This finishes the proof of the Proposition 3.

Proposition 4. *$u_i w_i$, $u_k w_k$ are not adjacent to $u_j w_j$.*

Proof. We need only to prove that $u_i w_i$ is not adjacent to $u_j w_j$. The proof that $u_k w_k$ is not adjacent to $u_j w_j$ is similar. Assume to the contrary that $u_i w_i$ is adjacent to $u_j w_j$. We have the following Claims (from Claim 16 to Claim 20):

Claim 16. *$u_i w_j \in E(G)$ and $w_i u_j \notin E(G)$. Moreover, $x_{i+1} \equiv x_j$.*

Proof. Assume to the contrary that $u_i w_j \notin E(G)$, then by the assumption that $u_i w_i$ is adjacent to $u_j w_j$ and by Lemma 6, $w_i u_j \in E(G)$. By Lemma 6 and by Proposition 3, $N(u_i) \subseteq X \cup \{w_i\}$. Arguing similarly to the proofs of Lemma 15, we have $x_i \equiv x_1$ and $x_{k+1} \equiv x_m$ and $x_{i+1} \equiv x_j$. Because of $|S| \geq 4$, $x_{j+1} \neq x_k$ and there are at least three 1-arcs on $x_{j+1}\overrightarrow{C}x_k$. We conclude that $u_t \in Y$ and $u_t \notin N(x_j)$ for any $j < t < k$, otherwise $P = (x_1 u_i w_i u_j\overrightarrow{C}u_t x_j)$ is a bad-path, a contradiction. Therefore $N(u_{j+1}, u_{j+2}) \subseteq X - \{x_j\}$ and $|N(u_{j+1}, u_{j+2})| \leq |X| - 1$. By $d(u_{j+1}, u_{j+2}) = 2$, $|N(u_{j+1}, u_{j+2})| \geq NC2(G) = |X|$, a contradiction.

Thus, $u_i w_j \in E(G)$. By Lemma 13, $w_i u_j \notin E(G)$. By Lemma 6 and by Proposition 3, $N(w_i) \subseteq X \cup \{u_i\}$. By Lemma 14, we have $x_{i+1} \equiv x_j$.

Claim 17. *$d(w_i, u_j) = 2$ and $x_i, x_{j+1} \notin N(w_i, u_j)$.*

Proof. By Claim 16, $d(w_i, u_j) = 2$. By Lemma 10, $x_i, x_{j+1} \notin N(w_i, u_j)$.

Claim 18.$x_j \notin N(v) \cap N(s_0)$.

Proof. Assume to the contrary that $x_j \in N(v) \cap N(s_0)$. Since Lemma 12, we have $x_1, x_m \notin N(w_i, u_j)$. By Lemma 6 and by Claim 17, $N(w_i, u_j) \subseteq X \cup \{u_i, w_j, w_k\} - \{x_1, x_m, x_i, x_{j+1}\}$. Because of $|N(w_i, u_j)| \geq NC2(G) = |X|$, we get $x_i \equiv x_1$ and $u_j w_k \in E(G)$.

Since Lemma 13, we have $u_k w_j \notin E(G)$, by Lemma 6 and by Proposition 3, $N(u_k) \subseteq X \cup \{w_k\}$. By Lemma 14, $x_{j+1} \equiv x_k$.

By $|S| \geq 4$, $x_{k+1} \neq x_m$ and there are at least three 1-arcs on $x_{k+1}\overrightarrow{C}x_m$. Since $|X| = NC2(G) \leq |N(w_i, u_j)| \leq |X \cup \{u_i, w_j, w_k\} - \{x_1, x_m, x_{j+1}\}| = |X|$, so $x_t \in N(w_i, u_j)$ for every $k + 1 < t < m$. Moreover, $u_t \in Y$ and $u_t x_{k+1} \notin E(G)$. Indeed, assume otherwise that $u_t x_{k+1} \in E(G)$, we distinguish two cases:

(1) If $x_t w_i \in E(G)$ then $P = (x_m\overleftarrow{C}u_t x_{k+1}\overrightarrow{C}x_t w_i u_i w_j\overrightarrow{C}w_k u_j x_j)$ is a bad-path, a contradiction.

(2) If $x_t u_j \in E(G)$ then $P = (x_m\overleftarrow{C}u_t x_{k+1}\overrightarrow{C}x_t u_j w_k\overleftarrow{C}w_j u_i w_i x_j)$ is a bad-path, a contradiction.

Clearly, $u_{k+2}, u_{k+3} \in Y$ and $N(u_{k+2}, u_{k+3}) \subseteq X - \{x_{k+1}\}$ and $d(u_{k+2}, u_{k+3}) = 2$, we conclude that $|X| = NC2(G) \leq |N(u_{k+2}, u_{k+3})| \leq |X - \{x_{k+1}\}| = |X| - 1$, a contradiction.

Note: Let $y \in \{v, s_0\}$ such that $y \notin N(x_j)$. Clearly that $d(y, u_1) = d(y, w_{m-1}) = 2$.

Claim 19. *If $x_i \neq x_1$ then $x_1 \notin N(w_i, u_j)$. Similarly, if $x_{k+1} \neq x_m$ then $x_m \notin N(w_i, u_j)$.*

Proof. Assume that $x_i \neq x_1$, then $u_1 \in Y$. If $u_1 x_j \notin E(G)$ then $N(y, u_1) \subseteq X - \{x_j\}$, so $|X| = NC2(G) \leq |N(y, u_1)| \leq |X| - 1$, a contradiction. Therefore, $u_1 x_j \in E(G)$. By Lemma 11, we have $x_1 \notin N(w_i, u_j)$. Similarly, if $x_{k+1} \neq x_m$ then $x_m \notin N(w_i, u_j)$.

Claim 20. *$x_i \equiv x_1$ and $x_{k+1} \equiv x_m$.*

Proof. Assume that $x_i \neq x_1$ and $x_{k+1} \neq x_m$. By Lemma 6 and by Proposition 3 and by Claims 16, 17, 19, we have $N(w_i, u_j) \subseteq X \cup \{u_i, w_j, w_k\} - \{x_1, x_m, x_i, x_{j+1}\}$, so $|X| = NC2(G) \leq |N(w_i, u_j)| \leq |X| - 1$, a contradiction. Thus, $x_i \equiv x_1$ or $x_{k+1} \equiv x_m$.

If $x_i \neq x_1$ then $x_{k+1} \equiv x_m$ and $N(w_i, u_j) \subseteq X \cup \{u_i, w_j, w_k\} - \{x_1, x_i, x_{j+1}\}$.

Because $|N(w_i, u_j)| \geq NC2(G) = |X|$, so $N(w_i, u_j) = X \cup \{u_i, w_j, w_k\} - \{x_1, x_i, x_{j+1}\}$ and by Proposition 3, $u_j w_k \in E(G)$. Since Lemma 13, $w_j u_k \notin E(G)$, so by Lemma 6 and by Proposition 3, $N(u_k) \subseteq X \cup \{w_k\}$. By Lemma 14, we have $x_{j+1} \equiv x_k$.

Because $|S| \geq 4$, there are at least three 1-arcs on $x_1\overrightarrow{C}x_i$. Clearly that $x_t \in N(w_i, u_j)$ for every $1 < t < i$.

Moreover, $w_{t-1} \in Y$ and $w_{t-1}x_i \notin E(G)$. Indeed, assume that $w_{t-1}x_i \in E(G)$. Then:

(1) If $x_t \in N(w_i)$ then $P = (x_1\overrightarrow{C}w_{t-1}x_i\overleftarrow{C}x_t w_i u_i w_j u_j x_j)$ is a bad-path, a contradiction.

(2) If $x_t \in N(u_j)$ then $P = (x_1\overrightarrow{C}w_{t-1}x_i\overleftarrow{C}x_t u_j w_j u_i w_i x_j)$ is a bad-path, a contradiction.

Therefore, $N(w_{i-2}, w_{i-3}) \subseteq X - \{x_i\}$ and $|N(w_{i-2}, w_{i-3})| \le |X| - 1$. Because $d(w_{i-2}, w_{i-3}) = 2$, this implies that $|N(w_{i-2}, w_{i-3})| \ge NC2(G) = |X|$, a contradiction.

Thus, $x_i \equiv x_1$. Similarly, we have $x_{k+1} \equiv x_m$.

By Claim 16 and by Claim 20, we conclude that $x_i \equiv x_1$, $x_{k+1} \equiv x_m$ and $x_{i+1} \equiv x_j$. By $|S| \ge 4$, $x_{j+1} \ne x_k$ and there are at least three 1-arcs on $x_{j+1}\overrightarrow{C}x_k$.

If $u_j w_k \in E(G)$ then by Lemma 13, $w_j u_k \notin E(G)$. By Lemma 6 and by Proposition 3, $N(u_k) \subseteq X \cup \{w_k\}$. By Lemma 14, $x_{j+1} \equiv x_k$, a contradiction. Therefore $u_j w_k \notin E(G)$. By Lemma 6 and by Proposition 3 and by Claim 17, we have $N(w_i, u_j) \subseteq X \cup \{u_i, w_j\} - \{x_1, x_{j+1}\}$.

Arguing similarly to the proofs of Claim 18, we have $u_t \in Y$ and $u_t x_{j+1} \notin E(G)$ for every $j+1 < t < k$. Therefore $N(u_{j+2}, u_{j+3}) \subseteq X - \{x_{j+1}\}$ and $|N(u_{j+2}, u_{j+3})| \le |X| - 1$, which contradicts to the fact that $|N(u_{j+2}, u_{j+3})| \ge NC2(G) = |X|$. This finishes the proof of the Proposition 4.

Finally, we conclude that $u_i w_i, u_j w_j, u_k w_k$ are pairwise non-adjacent arcs. By Lemma 6, $\omega(G - X) \ge |X| + 1$, which contradicts to the assumption that G is 1-tough.

Thus, the Case 2.2. does not happen.

IV. CONCLUSIONS

In this paper, we present some lower bounds for the circumference of tough graphs and prove the conjecture posed by Bauer, Fan and Veldman [1], namely, that: "*If G is 1-tough graph with $\sigma_3 \ge n$, then $c(G) \ge min\{n, 2NC2(G) + 4\}$*".

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