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# MVUE of Failure Rate for Exponential Class Software Reliability Models

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***Abstract***— ***Software quality has become a major concern of all software manufacturers. Quality of software can be measured using software reliability models, which express the behavior of failure with time. One such category of models is the exponential class software reliability models. In these models, the failure times are assumed to be exponentially distributed with a parameter Φ, called the failure rate. Estimation of this failure rate will help in the estimation of various measures of reliability of the given software. The parameter Φ can be estimated using various methods of estimation. The simplest among these methods, is the method of maximum likelihood estimation. Even though it satisfies most of the desirable properties of a good estimator, it is still not as efficient as the minimum variance unbiased estimator. In this paper, the minimum variance unbiased estimator of failure rate Φ for exponential class software reliability models is found. The procedure called Blackwellization is used to find this estimator. The efficiency of this estimator can be found by finding its variance and comparing it with the variance of already existing MLE. It is found that MVUE of failure rate has less variance than that of MLE of failure rate, thus indicating that MVUE of failure rate Φ is more efficient than already existing MLE of Φ.***

***Keywords***— ***Failure rate, Maximum likelihood estimation, Minimum variance unbiased estimator, Reliability, Software reliability models.***

## I. INTRODUCTION

Software quality is an important aspect in the field of software. Because of high competition among software products, customers prefer to procure software with high reliability. Software reliability models are the mathematical models, which express the behavior of failure with time. All these models are expressed in terms of certain number of parameters. Exponential class models are one such category of models. In these models, the failure times are assumed exponential and are expressed in terms of the failure rate Φ. To obtain the reliability measures of such models, it is necessary to estimate the value of the parameter Φ ([1]). There are several methods of estimation of parameters. The estimators are expected to satisfy certain desirable statistical properties. The method of Maximum Likelihood Estimation (MLE) is the widely used method of estimation. But, usually such an MLE is biased and is not efficient. To obtain an unbiased and more efficient estimator of failure rate Φ, method of Minimum Variance Unbiased Estimation is used.

### *A. Some terminologies used:*

Software reliability ([2]): It is the probability of failure-free operation of a computer program in a specified environment for a specified time.

Software reliability models ([2], [3]): These describe the behavior of failure with time, by expressing failures as random processes in either times of failure or the number of failures, at fixed times.

Exponential class models ([2], [3]): This group consists of all finite failure models the time of failure being exponential.

Estimator ([4], [5]): A function of the sample observations used to estimate the value of the unknown parameter of any distribution is said to be an estimator.

Characteristics of good estimators ([4, [5]): A good estimator should be (i) Consistent (ii) Unbiased (iii) Sufficient (iv) Efficient.

Consistency: An estimator T based on a sample of size n from a distribution with parameter θ is said to be consistent for θ, if T converges in probability to θ. i.e., if $P(|T-\theta|<\varepsilon)\rightarrow 1$ as $n\rightarrow\infty$, for every $\varepsilon>0$

Unbiasedness: An estimator T based on a sample of size n from a distribution with parameter θ, is said to be unbiased for θ, if $E(T)=\theta$.

Sufficiency: An estimator T is said to be a sufficient estimator for the parameter θ, if it contains all the information in the sample regarding that parameter. Or, in other words, an estimator T is sufficient for the parameter θ if the conditional distribution of the sample given T is independent of θ. Using the factorization theorem, a statistic T is sufficient for the parameter θ, if the likelihood function L can be expressed as $L=g(T, \theta).h(x)$, where in g, θ depends on x only through T and h(x) is independent of θ.

Efficiency: If in a class of estimators, there exists one whose variance is less than that of any such estimator, then it is called the most efficient estimator. Thus, if $T_1$ and $T_2$ are two estimators, then $T_1$ is more efficient than $T_2$, if $V(T_1)<V(T_2)$.

 

Complete estimator: Let T be an estimator and let h(T) be any function of T. Then, T is said to be the complete estimator, if E[h(T)=0] implies h(T)=0.

Complete sufficient estimator: A sufficient estimator which is also complete is called the complete sufficient estimator.

## II. METHODS OF ESTIMATION

In order to estimate the reliability, various methods of estimation are available. Some of these methods are – Method of maximum likelihood estimation (MLE), Method of least squares, Method of moments, Method of minimum variance unbiased estimation (MVUE) etc. All these estimators should satisfy the statistical properties described above. The one that satisfies most of these statistical properties is considered as the best estimator. Two methods of estimation viz, the method of MLE and the method of MVUE are explained below.

### *A. Method of Maximum Likelihood Estimation (MLE)([5], [6], [7]):*

Let $x_1$, $x_2$, $x_3$,......$x_n$ be a random sample of size n from a population with probability function f(x; θ). Then, the likelihood function is given by $L=\prod_{i=1}^{n} f(x,\theta)$ We have to find an estimator that maximizes the likelihood function. i.e., $\frac{\partial L}{\partial \theta}=0$ with $\frac{\partial^2 L}{\partial \theta^2}<0$ . Since L>0, we can write the above as, $\frac{1}{\mathrm{L}}\frac{\partial \mathrm{L}}{\partial \theta}=0$ i. e., $\frac{\partial(\ln \mathrm{L})}{\partial \theta}=0$. Thus, MLE is the solution of $\frac{\partial(\ln \mathrm{L})}{\partial \theta}=0$ with $\frac{\partial^2(\ln \mathrm{L})}{\partial \theta^2}<0$

### *B. Method of Minimum Variance Unbiased Estimation (MVUE)([5])*

If a statistic T based on a sample of size n is such that - T is unbiased for θ and has the smallest variance among the class of all unbiased estimators of θ, then, T is called the MVUE of θ. It is found that such an MVUE is always unique.

To find this MVUE, we start with an unbiased estimator U and then improve upon it by defining a function Φ(t) of the sufficient statistic. This procedure, called Blackwellization is explained below:

Let $U=U(x_1,x_2,....x_n)$ be an unbiased estimator of parameter θ and let $T=T(x_1, x_2,....x_n)$ be a sufficient statistic for θ. Consider the function Φ(t) of the sufficient statistic defined as Φ(t)= E[U|T=t)]. Then, E[Φ(t)]=θ and V[Φ(t)]≤ V(U). If in addition, T is also complete, then Φ(t) is also the unique estimator.

The MVUE as obtained using the method of Blackwellization always gives an estimator that is more efficient than any other estimator. That means it always gives an estimator that has less variance than any other estimator.

## III. MLE AND MVUE OF FAILURE RATE

.MLE is a very simple and widely used method of estimation. MLEs are always consistent and sufficient. But they need not be unbiased and efficient. But, MVUEs are always unbiased and efficient. Thus they are considered as the best estimators among the class of all estimators.

Let us consider the problem of finding MLE and MVUE of failure rate Φ and compare their measures of dispersion, viz, the variance, to find the best among the two.

***A.*** MLE of Φ: In the exponential class models, it is assumed that the time to failure of individual fault is exponentially distributed with parameter Φ, the failure rate. Thus, if T denotes the time to failure, then T~ exp(Φ). Hence, its probability function is given by $f(t)=\Phi e^{-\Phi t}$ . The likelihood function is given by

$L=\prod_{i=1}^{n} f(t_i;\Phi)=\Phi^n e^{-\Phi\sum_{i=1}^{n} t_i}$ -----------(1). Using the procedure to find MLE, we get the MLE of Φ as the solution of $\frac{\partial \ln L}{\partial \Phi}=0$ , which gives $\frac{n}{\Phi}-\sum_{i=1}^{n} t_i=0.$ This gives the MLE of Φ as $\hat{\Phi}=\frac{n}{\sum_{i=1}^{n} t_i}$ .

***B.*** MVUE of Φ: To find MVUE of Φ, we have to find the unbiased estimator of Φ, say U and then a complete sufficient estimator say T. The MVUE of Φ is then found as E(U|T).

To find the unbiased estimator of Φ: Let $P=\sum_{i=1}^{n} t_i$ . Then, since each $T_i$ ~ exp (Φ), P~G (n, Φ). Hence, its probability function is given by $f(p)=\frac{\Phi^n}{\Gamma n} e^{-\Phi p} p^{n-1}$ where p>0, n>0. From this, we get $E(P)=\int_0^\infty t\frac{\Phi^n}{\Gamma n} e^{-t\Phi} t^{n-1} dt=\frac{n}{\Phi}$ . Also, $E(\frac{1}{P})=\int_0^\infty \frac{1}{t}\frac{\Phi^n}{\Gamma n} e^{-t\Phi} t^{n-1} dt=\frac{\Phi^n}{\Gamma n}\frac{\Gamma(n-1)}{\Phi^{n-1}}=\frac{\Phi}{n-1}$ . From this, we get $E\left(\frac{n-1}{P}\right)=\Phi.$ Thus, $\frac{n-1}{P}=\frac{n-1}{\sum_{i=1}^{n} t_i}$ is unbiased for Φ.

To find the complete sufficient estimator: The factorization theorem due to Fisher and Neymann says that a statistic T(x) is sufficient for the parameter Φ if the likelihood function L(x) can be expressed as L=g(T(x), Φ).h(x), where g depends on Φ and x only through T(x) and h(x) is independent of Φ. Applying this factorization theorem to (1), we get $T=\sum_{i=1}^{n} t_i$ as the sufficient estimator of Φ. Using the definition of complete sufficient estimator mentioned in the introduction section, we get $T=\sum_{i=1}^{n} t_i$ as the complete sufficient estimator. Since unbiased estimator is a function of the complete sufficient estimator, it is the MVUE. Thus, MVUE of Φ is given by $\tilde{\Phi}=\frac{n-1}{\sum_{i=1}^{n} t_i}$

## IV. COMPARISON

Let us compare the two estimators by comparing their variances.

*A. Variance of MLE of Φ:* MLE as obtained above is $\hat{\Phi}=\frac{\mathrm{n}}{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{t_i}}$. To find its variance, first let us find its distribution. For this, consider the transformation $Y=\frac{n}{T}$ where $T=\sum_{i=1}^{n} t_i$. Using the concept of functions of random variables, the probability density function of y= $\hat{\Phi}$ is given by $g(y)=f(t)\left|\frac{dt}{dy}\right|$ where t is expressed in terns of y. Using the distribution of T, we get $g(y)=\frac{\Phi^n}{\Gamma n}e^{-\frac{n\Phi}{y}}\left(\frac{n}{y}\right)^{n-1}\frac{n^2}{y^2 n}$. This simplifies to $g(y)=\frac{(\Phi n)^n}{\Gamma n}\frac{1}{y^{n+1}}e^{-\frac{n\Phi}{y}}$ where y>0. To find the variance of $\hat{\Phi}$, we need to find E(Y) and V(Y). Consider $E(Y)=\int_0^\infty y g(y)dy=\int_0^\infty y\frac{(\Phi n)^n}{\Gamma n}e^{-\frac{n\Phi}{y}}\frac{1}{y^{n+1}}dy=\frac{n\Phi}{n-1}$ (2).

Similarly, $E(Y^2)=\int_0^\infty y^2 g(y)dy=\int_0^\infty y^2\frac{(\Phi n)^n}{\Gamma n}e^{-\frac{n\Phi}{y}}\frac{1}{y^{n+1}}dy=\frac{n^2\Phi^2}{(n-1)(n-2)}$. Thus, the variance of y= $\hat{\Phi}$ is given by $V(Y)=E(Y^2)-(E(Y))^2=\frac{n^2\Phi^2}{(n-1)(n-2)}-\left(\frac{n\Phi}{n-1}\right)^2$. Hence $V(Y)=\frac{n^2\Phi^2}{(n-1)^2(n-2)}$.

That is $V(\hat{\Phi})=\frac{n^2\Phi^2}{(n-1)^2(n-2)}$ (3)

*B. Variance of MVUE of Φ:* MVUE as obtained above is $\tilde{\Phi}=\frac{n-1}{\sum_{i=1}^{n} t_i}$. Let us find its variance by finding its distribution first. For this, consider the transformation $U=\frac{n-1}{T}$ where $T=\sum_{i=1}^{n} t_i$. Again using the concept of functions of random variables, probability density function of u= $\tilde{\Phi}$ is given by $h(u)=f(t)\left|\frac{dt}{du}\right|$ where t is expressed in terms of u. Since the distribution of T is already known, we get $h(u)=\frac{\Phi^n}{\Gamma n}\frac{(n-1)^n}{u^{n+1}}e^{-\frac{(n-1)\Phi}{u}}$ where u>0.

We have from (2), $E(Y)=\frac{n\Phi}{n-1}$. From this, we get $E\left(\frac{(n-1)Y}{n}\right)=\Phi$. Since $Y=\frac{n}{T}$, we have, $E\left(\frac{(n-1)}{T}\right)=\Phi$. i.e., $\frac{n-1}{T}$ is unbiased for Φ. i.e., $\frac{n-1}{\sum_{i=1}^{n} t_i}$ is unbiased for Φ. i.e. $\tilde{\Phi}$ is unbiased for Φ.

Since $U=\tilde{\Phi}$ is unbiased for Φ, we have, E(U)= Φ. Also, $E(U^2)=\int_0^\infty u^2 h(u)du=\int_0^\infty u^2\frac{\Phi^n}{\Gamma n}e^{-\frac{(n-1)u}{\Phi}}\frac{(n-1)^n}{u^{n+1}}du$.

i.e., $E(U^2)=\Phi^2\frac{(n-1)}{(n-2)}$. Thus the variance of u= $\tilde{\Phi}$ is given by $V(U)=E(U^2)-(E(U))^2=\Phi^2\frac{(n-1)}{(n-2)}-\Phi^2$.

 

Simplifying this, we get $V(U) = \dfrac{\Phi^2}{(n-2)}$. That is, $V(\tilde{\Phi}) = \dfrac{\Phi^2}{(n-2)}$ (4).

Let us compare the variance of mle and mvue of Φ.

Consider $V(\hat{\Phi}) = \dfrac{n^2\Phi^2}{(n-1)^2(n-2)} = \dfrac{\Phi^2}{\left(\frac{n-1}{n}\right)^2(n-2)} = \dfrac{\Phi^2}{\left(1-\frac{1}{n}\right)^2(n-2)}$

i.e., $V(\hat{\Phi}) = \dfrac{V(\tilde{\Phi})}{\left(1-\frac{1}{n}\right)^2}$. i.e., $V(\tilde{\Phi}) = V(\hat{\Phi})\left(1-\frac{1}{n}\right)^2 \le V(\hat{\Phi})$ (5).

## V. CONCLUSIONS

There are various methods of estimation and the method of MLE is widely used because of its simplicity. However, it does not satisfy all the desirable properties of good estimators. If we have a set of estimators, the one with less variance is always considered as the best estimator as given by the efficiency property given earlier. As obtained above, the MVUE of Φ provides a variance which is less than that given by MLE of Φ. i.e., $V(\tilde{\Phi}) \le V(\hat{\Phi})$. Thus, using the efficiency property of estimators, we can conclude that $\tilde{\Phi}$ is more efficient than $\hat{\Phi}$. Since failure rate plays an important role in estimating various measures of reliability of software, the MVUE of Φ obtained above is used to estimate various measures of software reliability of exponential models more accurately.

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