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**Research Paper** 

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# Directional q-frame Along a Space Curve

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Abstract— We consider a curve in Euclidean 3-Space and the directional q-frame along the curve. The basic idea behind the directional q-frame is that the quasi-normal vector is the cross product of the tangent vector with the fixed vector. In this paper, we establish the local theory of space curves according to the directional q-frame. Moreover, we show the advantages of q-frame over the other frames such as the Frenet frame and Bishop (rotation-minimizing) frame.

Keywords— curve framing, Frenet frame, parallel transport frame, Bishop frame, space curve.

# I. INTRODUCTION

Bishop showed that we can define lots of frame along a space curve apart from the Frenet frame [1, 2]. Inspired by the 3D offset curve application of the quasi-normal vector  $\mathbf{n}_{a}$  [5, 6], Dede et al. (2015) introduced the directional q-frame along a space curve to construct a tubular surface [13]. The directional q-frame offers two key advantages over the Frenet frame [3, 8]: a) it is well defined even if the curve has vanishing second derivative [11], b) it avoid the unnecessary twist around the tangent. Moreover, the computation of the directional q-frame is easier than the rotation minimizing frames [4, 9, 10, 12], one of them is Bishop frame [1, 2].

The directional q-frame of a regular curve  $\alpha(t)$  is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q$$
(1)

where  $\mathbf{k}$  is the projection vector.

The variation equations of the directional q-frame is given by

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_{q} \\ \mathbf{b}_{q} \end{bmatrix} = \|\boldsymbol{\alpha}'\| \begin{bmatrix} 0 & k_{1} & k_{2} \\ -k_{1} & 0 & k_{3} \\ -k_{2} & -k_{3} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_{q} \\ \mathbf{b}_{q} \end{bmatrix}$$
(2)

where the q-curvatures are expressed as follows

$$k_{1} = \frac{\langle \mathbf{t}', \mathbf{n}_{q} \rangle}{\|\boldsymbol{\alpha}'\|}, k_{2} = \frac{\langle \mathbf{t}', \mathbf{b}_{q} \rangle}{\|\boldsymbol{\alpha}'\|}, k_{3} = -\frac{\langle \mathbf{n}_{q}, \mathbf{b}'_{q} \rangle}{\|\boldsymbol{\alpha}'\|}.$$
(3)

# **II. LOCAL THEORY OF SPACE CURVE**

In this chapter, we will begin an investigation into the local theory of space curve by using the directional q-frame. First, we establish the invariance of the q-curvatures under an Euclidean motion of  $\Re^3$ . Then, we classified the directional q-frame into three types.

**Theorem 2.1.** Let  $\alpha(t)$  be a regular curve. Then the q-curvatures are given by

$$k_{1} = \frac{\det[\alpha'', \alpha', \mathbf{k}]}{\|\alpha'\|^{2} \|\alpha' \wedge \mathbf{k}\|}, k_{2} = \frac{\langle \alpha', \mathbf{k} \rangle \langle \alpha'', \alpha' \rangle - \|\alpha'\|^{2} \langle \alpha'', \mathbf{k} \rangle}{\|\alpha'\|^{3} \|\alpha' \wedge \mathbf{k}\|}$$

and

$$k_{3} = \frac{\langle \alpha', \mathbf{k} \rangle \det[\alpha', \alpha'', \mathbf{k}]}{\|\alpha' \wedge \mathbf{k}\|^{2} \|\alpha'\|^{2}}.$$

**Proof:** Using (1) yields that  $\alpha'(t) = \|\alpha'\| \mathbf{t}$  then differentiating and substituting (2) into the result gives

$$\boldsymbol{\alpha}''(s) = \|\boldsymbol{\alpha}'\| \mathbf{t} + \|\boldsymbol{\alpha}'\|^2 k_1 \mathbf{n}_q + \|\boldsymbol{\alpha}'\|^2 k_2 \mathbf{b}_q.$$

On the other hand, from (1) we have  $\alpha' \wedge \mathbf{k} = |\alpha' \wedge \mathbf{k}| |\mathbf{n}_{\alpha}|$ . Hence, together with the above equation we have

$$\langle \alpha''(s), \alpha' \wedge \mathbf{k} \rangle = \| \alpha' \wedge \mathbf{k} \| \| \alpha' \|^2 k_1.$$

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Thus

$$k_1 = \frac{\det[\alpha'', \alpha', \mathbf{k}]}{\|\alpha'\|^2 \|\alpha' \wedge \mathbf{k}\|}$$

Differentiating the tangent vector  $\mathbf{t}$  gives

 $\mathbf{t}' = \frac{\alpha'' \|\alpha'\| - \alpha' \|\alpha'\|'}{\|\alpha'\|^2}.$ (4)

Substituting (1) and (4) into (3), we then have

$$k_{2} = \frac{1}{\left\|\boldsymbol{\alpha}'\right\|^{3}} \left\langle \boldsymbol{\alpha}'', \boldsymbol{\alpha}' \wedge \frac{\boldsymbol{\alpha}' \wedge \mathbf{k}}{\left\|\boldsymbol{\alpha}' \wedge \mathbf{k}\right\|} \right\rangle$$

Using Lagrange's formula yields

$$k_{2} = \frac{\langle \boldsymbol{\alpha}', \mathbf{k} \rangle \langle \boldsymbol{\alpha}'', \boldsymbol{\alpha}' \rangle - \left\| \boldsymbol{\alpha}' \right\|^{2} \langle \boldsymbol{\alpha}'', \mathbf{k} \rangle}{\left\| \boldsymbol{\alpha}' \right\|^{3} \left\| \boldsymbol{\alpha}' \wedge \mathbf{k} \right\|}$$

Similarly,

$$k_{3} = \frac{\langle \alpha', \mathbf{k} \rangle \det[\alpha', \alpha'', \mathbf{k}]}{\|\alpha' \wedge \mathbf{k}\|^{2} \|\alpha'\|^{2}}.$$

**Corollary 2.2.** It is easy to see that the q-curvatures  $k_1, k_2$  and  $k_3$  depend on the projection vector **k**. Thus, we state the following theorem.

**Theorem 2.3.** Let  $\alpha$  be a regular curve with the projection vector  $\mathbf{k}$  and q-curvatures  $k_1, k_2$  and  $k_3$ . Let  $F : \mathfrak{R}^3 \to \mathfrak{R}^3$  be an Euclidean motion with the linear part A. Then, the q-curvatures are invariant under the Euclidean motion if the curve  $\gamma = F \circ \alpha$  has the new projection vector  $\mathbf{k}^* = A\mathbf{k}$ .

**Proof:** The curve  $\gamma$  can be written as

$$\gamma(t) = A\alpha(t) + F(0)$$

with the q-curvatures  $k_1^*, k_2^*$  and  $k_3^*$ . By differentiating the above equation, we have  $\gamma'(t) = A\alpha'(t)$  and  $\gamma''(t) = A\alpha''(t)$ . Thus, from (3) it follows that

$$k_{1}^{*} = \frac{\det[\gamma'', \gamma', \mathbf{k}^{*}]}{\|\gamma'\|^{2} \|\gamma' \wedge \mathbf{k}^{*}\|} = \frac{\det[A\alpha'', A\alpha', A\mathbf{k}]}{\|A\alpha'\|^{2} \|A\alpha' \wedge A\mathbf{k}\|}.$$
(5)

On the other hand, it is easy to see that

$$\|A\alpha'\| = \|\alpha'\|, \|A\mathbf{k}\| = \|\mathbf{k}\|, \|A\alpha' \wedge A\mathbf{k}\| = \|\alpha' \wedge \mathbf{k}\|.$$

Substituting the above equation into (5) gives

$$k_1^* = \frac{\det[\alpha'', \alpha', \mathbf{k}]}{\|\alpha'\|^2 \|\alpha' \wedge \mathbf{k}\|} = k_1$$

The other q-curvatures  $k_2^*$  and  $k_3^*$  can be obtained by using the similar method. In order to have the directional q-frame to be Euclidean invariant, it is necessary that the projection vector **k** and the axis of Euclidean motion must be identical. Thus, we classified the q-frame into three types: *z*-axis, *y*-axis and *x*-axis directional q-frames denoted by  $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}_z\}$ ,  $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}_y\}$  and  $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}_x\}$  with the projection vectors  $\mathbf{k}_z = (0, 0, 1)$ ,  $\mathbf{k}_y = (0, 1, 0)$  and  $\mathbf{k}_x = (1, 0, 0)$ , respectively.



Fig. 1 y-axis directional q-frame along the line. The quasi-normal (red) and the quasi-binormal (black) vectors are shown.

Example 2.4. Now, let us consider the curve(line) parametrized by

$$\alpha(t) = (t, 1, 1)$$

From (1), the *y*-axis directional q-frame is obtained as follows:  $\mathbf{t} = (1,0,0), \mathbf{n}_q = (0,0,1)$  and  $\mathbf{b}_q = (0,-1,0)$  with  $\mathbf{k}_y = (0,1,0)$ . The *y*-axis directional q-frame is illustrated in Figure 1.

Example 2.5. Let us consider the curve given by

$$\alpha(t) = (\cos(t), \sin(t), t^3).$$

We have the *z*-axis directional q-frame in the following form

$$\mathbf{t} = \frac{1}{\sqrt{1+9t^2}} (-\sin(t), \cos(t), 3t^2)$$
  

$$\mathbf{n}_q = (\cos(t), \sin(t), 0)$$
  

$$\mathbf{b}_q = \frac{1}{\sqrt{1+9t^2}} (-3t^2 \sin(t), 3t^2 \cos(t), -\sqrt{1+9t^2})$$

where  $\mathbf{k}_{z} = (0, 0, 1)$ .

From (3), the q-curvatures are obtained as

$$k_1 = -\frac{1}{\sqrt{1+9t^2}}, k_2 = -\frac{6t}{1+9t^2}, k_3 = \frac{3t^2}{\sqrt{1+9t^2}}$$

Figure 2 compares the Frenet frame and z-axis directional q-frame vectors along the curve  $\alpha(t)$ .



Fig. 2 The Frenet frame (left) and the q-frame (right) along the curve. The normal-plane vectors are shown.

**Theorem 2.6.** The Darboux vector  $\mathbf{d}_q$  of the directional q-frame which is called the quasi-Darboux vector is obtained as

$$\mathbf{d}_q = k_3 \mathbf{t} - k_2 \mathbf{n}_q + k_1 \mathbf{b}_q$$

**Proof:** The variation of directional q-frame  $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$  along a curve is specified in terms of its vector angular velocity  $\mathbf{d}_q$  as

$$\mathbf{t}' = \mathbf{d}_q \wedge \mathbf{t}, \mathbf{n}_q' = \mathbf{d}_q \wedge \mathbf{n}_q, \mathbf{b}_q' = \mathbf{d}_q \wedge \mathbf{b}_q.$$
(6)

On the other hand,  $\mathbf{d}_{q}$  can be written as

$$\mathbf{d}_{q} = a\mathbf{t} + b\mathbf{n}_{q} + c\mathbf{b}_{q} \tag{7}$$

where  $a, b, c \in \mathfrak{R}$ .

Combining (2), (6) and (7) yields that

$$\mathbf{d}_q = k_3 \mathbf{t} - k_2 \mathbf{n}_q + k_1 \mathbf{b}_q.$$

Thus, the instantaneous angular speed of the directional q-frame is obtained by

$$\left\| \mathbf{d}_{q} \right\| = \sqrt{k_{1}^{2} + k_{2}^{2} + k_{3}^{2}}$$

The characteristic property of a rotation-minimizing frame(RMF) is that its angular velocity has no component along  $\mathbf{t}$ , since the directional q-frame is not RMF. However, let us consider again the curve in Example 2.5, the instantaneous angular speed of the *z*-axis directional q-frame is obtained by

$$\left\|\mathbf{d}_{q}\right\| = \sqrt{\frac{81t^{8} + 18t^{4} + 36t^{2} + 1}{(1+9t^{4})^{2}}}.$$

In fact, Figure 3 shows that despite the fact that the directional q-frame is not RMF, the behavior of the directional q-frame is similar to that of the RMF.

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Fig. 3 Comparison of instantaneous rates of rotation for the z-axis directional q-frame (a), the Bishop Frame (b) and the Frenet frame (c).

Example 2.7. Let us consider the curve given by

$$\alpha(t) = (\sqrt[3]{t^5} + 100, t^2 + t + 5, -t^3)$$

It is easy to see that the second derivative of the curve vanishes at t = 0. Thus, the Frenet frame can not be computed at this point (highlighted by an arrow in Figure 4). On the other hand, the z-axis directional q-frame can be determined at this point



Fig. 4 The Frenet frame (left) and the q-frame (right) along the curve. The normal-plane vectors are shown.

**Example 2.8**. So far we have tried to show that the directional q-frame is superior to other frames. Now it is convenient to think that which directional q-frame is better. In order to find the better one, we again trust the comparison of the instantaneous rates of rotation.

Let us consider a Bezier curve with the control points  $[0 \ 5 \ 0; 2 \ -2 \ 2; 2 \ -1 \ 1; 4 \ 10 \ 3]$ , the instantaneous rates of rotation for the *z*-axis, *y*-axis and *x*-axis directional q-frame are illustrated in Figure 5. Thus, *y*-axis directional q-frame of the Bezier curve shown in Figure 5.



Fig. 5 The comparison of instantaneous rates of rotation for *z*-axis (a), *x*-axis (c) and *y*-axis (b) directional q-frames (left) and the *y*-axis directional q-frame (right) along the curve. The normal-plane vectors are shown.

**Corollary 2.9.** Let  $\alpha$  be a line in  $\Re^3$ . Then the q-curvatures vanish identically,  $k_1 = k_2 = k_3 = 0$ . **Theorem 2.10.** Let  $\alpha$  be a unit speed curve. Then,  $\alpha$  is a plane curve if we have the following relation  $k_1k_2' - k_1'k_2 + k_2^2k_3 + k_1^2k_3 = 0.$ 

**Proof:** Assume that the curve  $\alpha$  lies in a plane which passes through the point p and is perpendicular to the unit vector u. Then we have

$$\langle \alpha - p, u \rangle = 0. \tag{8}$$

Differentiating (8) gives  $\langle \mathbf{t}, u \rangle = 0$ . Thus, differentiating this equation again, then substituting (2) into the result gives

$$k_1 \langle \mathbf{n}_q, u \rangle + k_2 \langle \mathbf{b}_q, u \rangle = 0.$$
<sup>(9)</sup>

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By differentiating (9), we have

$$(k_1^{'} - k_2 k_3) \langle \mathbf{n}_q, u \rangle + (k_1 k_3 + k_2^{'}) \langle \mathbf{b}_q, u \rangle = 0.$$
(10)  
on  $\langle \mathbf{n}_q, u \rangle$  and  $\langle \mathbf{b}_q, u \rangle$ . Solving (9) and (10) together gives

Thus, we get a systems of equations on  $\langle \mathbf{n}_q, u \rangle$  and  $\langle \mathbf{b}_q, u \rangle$ . Solving (9) and (10) together gives

$$k_1k_2' - k_1'k_2 + k_2^2k_3 + k_1^2k_3 = 0$$

which completes the proof.

#### **III. SPECIAL CASES**

In this section, we investigate some special cases. It is easy to see that the q-frame is not singular as often as the Frenet frame. However, there are some special cases need to be discussed.

**Case (i)** One of the directional q-frame can be singular when **t** and **k** are parallel. Observe that this case may occur when the curve is a line. For instance, *x*-axis directional q-frame is singular if the curve(line) is given by  $\alpha(t) = (x(t), a, b)$  with  $a, b \in \Re$ . To determine the *x*-axis directional q-frame along this curve, we consider the other directional q-frames. Let us denote by  $\mathbf{n}_{q,z}$ ,  $\mathbf{n}_{q,y}$  and  $\mathbf{n}_{q,x}$  the quasi-normal vectors of the *z*-axis, *y*-axis and *x*-axis directional q-frames, respectively. It is obvious that  $\mathbf{n}_{q,x}$  may be deduced from  $\mathbf{n}_{q,y}$  and  $\mathbf{n}_{q,z}$ . Thus we have

$$\mathbf{n}_{q,x} = \cos \varphi \mathbf{n}_{q,y} + \sin \varphi \mathbf{n}_{q,z}.$$

From (1) and above equation, the quasi-binormal vector  $\mathbf{b}_{a,x}$  is obtained as

$$\mathbf{p}_{q,x} = \sin \varphi \mathbf{b}_{q,y} + \cos \varphi \mathbf{b}_{q,z}.$$

where  $\varphi$  is the angle between  $\mathbf{n}_{q,x}$  and  $\mathbf{n}_{q,y}$ .

Let us consider the line in Example 1. The tangent vector  $\mathbf{t} = (1,0,0)$  and  $\mathbf{k}_x = (1,0,0)$  are identical. For  $\varphi = 3\pi/4$ , the quasi-normal vectors  $\mathbf{n}_{q,z}(red)$ ,  $\mathbf{n}_{q,y}(black)$  and  $\mathbf{n}_{q,x}(green)$  are shown in Figure 6. An analogous system can be formulated in the cases where y-axis and z-axis directional q-frames are singular.



Fig. 6 The quasi-normal vectors along the line .

**Case (ii)** The quasi-normal and the quasi-binormal vectors of *z*-axis and *x*-axis directional q-frames are in the opposite direction with the normal and the binormal vectors of Frenet frame when the curve is given by  $\alpha(t) = (x(t), y(t), a)$  or  $\alpha(t) = (a, y(t), z(t))$ , respectively. It is easy to see that we have  $\mathbf{n}_q = -\mathbf{n}, \mathbf{b}_q = -\mathbf{b}$  and  $k_1 = -\kappa, k_2 = 0, k_3 = \tau = 0.$ 

On the other hand, if the curve is parametrized by  $\alpha(t) = (x(t), a, z(t))$ , then *y*-axis directional q-frame is identical with the Frenet frame, namely,  $\mathbf{n}_q = \mathbf{n}, \mathbf{b}_q = \mathbf{b}$  and  $k_1 = \kappa, k_2 = 0, k_3 = \tau = 0$ .

## **IV. CONCLUSIONS**

Adapted frames on (primarily) spatial curves have had a renewed interest over the last ten years or so, mainly because of their applications on sweep surfaces and rotation-minimizing frame curves. In this paper, we have investigated the directional q-frame along a space curve. We have given some examples to illustrate the advantages of the proposed method. The main advantage of the directional q-frame is that it can be easily constructed even when the curvature of the curve vanishes.

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