Volume 5, Issue 11, November 2015 ISSN: 2277 128X
International Journal of Advanced Research in Computer Science and Software Engineering
Research Paper
Available online at: www.ijarcsse.com

g-Pre Regular And g-Pre Normal Topological Spaces

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Abstract - In general topology, the notion of pre-open set, introduced by A.S. Mashour et al. [1982], has a significant role and the most important generalizations of regularity & normality appear as the notions of pre-regularity along with strong regularity [1983] and pre-normality as well as strong normality [1984] respectively.

In 1970, N. Levine projected the concept of so called g-closed sets in topological spaces in an independent way and studied its basic properties.Since then many modifications of g-closed sets were defined and investigated by a large number of topologists. In 1996, Maki et al. introduced the concepts of gp-closed sets. The purpose of this paper is to study the classes of regular spaces & normal spaces, namely gp-regular spaces & gp-normal spaces which are a generalization of the classes of p-regular & p-normal spaces respectively. The paper also contains the behaviour of pre * —$T_{1/2}$ spaces whenever it is strongly regular or strongly normal. It also highlights the pre-topological property of a gp-normal pre- R_0 spaces.

Also, through this paper, a tribute is being paid to the renowned mathematician Professor M.E. Abd. El – Monsef who left for his heavenly abode on 13th August, 2014.

Keywords - g-pre –regular spaces, g-pre –normal spaces, $g^*\mathcal{K}$ – regular spaces, $g^*\mathcal{K}$ - normal spaces.

I. INTRODUCTION & PRELIMINARIES

Various new topological concepts& their basic properties have been defied & investigated using the notion of pre-open sets & pre-open, pre-continuous mappings(i.e. pre homeomorphism) as introduced by A.S. Mashhour et al. [1]. In 1998, T. Noiri et al. [2] studied generalized pre closed functions using generalized preclosed sets.

A subset A of a space (X, T) is known as a generalized pre-closed iff every open superset of A contains its pre-closure[2].

In the present paper, spaces(X,T) and (Y,σ) always mean topological spaces which are not assumed to satisfy any separation axioms are assumed unless explicitly mentioned.

Also, f: (X,T)→ (Y,σ) denotes a single valued function f of a space (X,T) into another space (Y,σ).And for a subset A of a space (X,T), X/A = A^c , cl(A) & int(A) denote the complement, the closure & the interior of A in (X,T) respectively

The concepts of g-pre –regular & g-pre –normal spaces are, here, studied using generalized- pre- closed sets.

Definition (1.1): A subset A of a topological space (X,T) is called

I. regular open or open domain[1] if A = int(cl(A)).
II. an α-open[3] set if A⊆ int(cl(int(A)))
III. pre-open [4] or nearly open[1] set if A ⊆ int(cl (A))

The compliments of the above mentioned open sets are their respective closed sets. The smallest $\mathcal{K}$-closed set containing A is called $\mathcal{K}$cl(A) where $\mathcal{K}$ = regular, α, p, s,β & π. The largest $\mathcal{K}$-open set contained in A is called $\mathcal{K}$int(A) where $\mathcal{K}$ = regular, α, p, s, β & π.

The family of all $\mathcal{K}$-open (resp. $\mathcal{K}$ -closed) sets of a space (X,T) is denoted by $\mathcal{K}$O(X)(resp. $\mathcal{K}$C(X)); here and above $\mathcal{K}$ = regular, α, p, s, β & π.

Definition(1.2)[1,5]: (a) The p-interior of a subset A, denoted as p-int(A), is defined as the union of all p-open sets contained in A.

(b) The pre - closure of subset A, denoted as p - cl(A), is defined as the intersection of all p - closed sets containing A. Naturally, pint(A) is pre open where as pcl(A) is pre closed where A is subset of X.

Also, pint(A)= A ∩(int(cl(A)) &
pcl(A)= A∪ cl(int (A)).

Definition (1.3)[2]: A subset A of a space (X,T) is said to be generalized preclosed (briefly gp-closed) iff pcl(A) ⊆ U whenever A ⊆ U and U is open in X.

Definition (1.4)[2]: For a subset A of a space (X,T)
gpcl(A) = ∩ { A⊂ F , F is gp-closed in X} & gpint(A) = ∪ { V⊂ A, V is gp-open in X}
also, gpcl(A) ⊂ pcl(A) as every pre closed set is gp- closed.

Definition (1.5)[6]: A space (X,T) is called pre * —$T_{1/2}$ if every gp-closed set in X is preclosed.

Definition (1.6): A function f:(X,T)→(Y,σ) is called pre continuous [1] (resp. pre irresolute[7]) if the inverse image of each open(resp. pre open) set of Y is pre open in X.

Definition (1.7)[8]: A bijective function f : (X,T)→(Y,σ) is called pre- homeomophism if f is M-pre open and pre-irresolute.

Definition (1.8) [5]: A space (X,T) is called strongly regular if for each pre closed set A & each point x ∉ A , there exist pre-open sets U & V such that x∈ U & A ⊂ V.

Definition (1.9)[9]: A space (X,T) is called strongly normal if for each pair of disjoint preclosed sets A & B, there exist pre-open sets U & V such that A ⊆U & B ⊆ V.

Any other notion and symbol, not defined in this paper, may be found in the appropriate reference.

II. g- PRE REGULAR SPACES

This section introduces g-pre regular spaces in topological spaces.

Definition (2.1): A topological space (X,T) is said to be g-pre-regular (in short gp-regular) space iff every gp-closed set F and every point x ∉ F , there exist disjoint pre-open sets U & V such that F ⊂ U & x∈ V .Obviously, every gp-regular space is strongly regular space, but not conversely.

Lemma (2.2): A strongly regular pre *—$T_{1/2}$ space is gp-regular.

Proof: Let (X,T) be a strongly regular space as well as pre *—$T_{1/2}$ space. Since,(X,T) is a pre *—$T_{1/2}$ space, hence every gp-closed set in X is pre closed i.e. the class of gp-closed sets & pre-closed sets coincide. Now, (X, T) is strongly regular space which provides that for each pre closed set A & each point x ∉ A , there exist disjoint pre-open sets U & V such that x∈ U & A ⊂ V. Combining these facts, it is concluded that for each gp-closed set A and each point there exist disjoint pre-open sets U & V such that A ⊂ U & x∈ V , which turns (X,T) to be a gp-regular.

Characterization criteria:

Theorem (2.3): A topological space (X,T) is gp- regular iff every gp-closed set F and every point x ∉ F,there exists pre-open sets U & V such that x∈ U, F ⊂ V and pcl(U) ∩ pcl(V) = φ.

Proof: Suppose that F is a gp- closed set of a space (X, T) and x ∉ F. Since, (X,T) is a gp-regular space hence, there exist disjoint pre-open sets U & V such that F ⊂ V & x∈ U & U∩ V = φ. Obviously, U∩ V = φ ⇒ U∩ pcl(V)= φ& pcl(U)∩ V = φ⇒ pcl(U)∩ pcl(V) = φ. Converse is not natural, so omitted.

Theorem (2.4): For a space (X, T) the following are equivalent:

(i) (X, T) is gp-regular.

(ii) for every x∈ X and for every gp- open set W containing x there exists a pre open set V such that pcl V⊆ W.

(iii) for every gp-closed set F and every point x ∉ F , there exists pre-open sets V such that pcl(V) ∩ F = φ.

Proof: (i) ⇒ (ii):

Let (X,T) be a gp-regular space. Let W be a gp-open set containing a point x ∈ X. Since, W^C is gp-closed set and x ∉ W^C, hence by the hypothesis, there exist pre-open sets U & V such that W^C ⊆ U , x ∈ V and U ∩ V = φ.

Now, U ∩ V = φ ⇒ V ⊆ U^C

⇒ pcl(V) ⊆ pcl(U^C) = U^C

Again, W^C ⊆ V ⇒ U^C ⊆ W.

Combining these two relations, we get pcl(V) ⊆ W.

(ii) ⇒ (i):

Let F be any gp-closed set and x ∉ F. Then x ∈ F^C and F^C is a gp-open set. By hypothesis, there exists a pre-open set V of x such that pcl(V) ⊆ F^C, obviously, F⊆ $(pcl(V))^C$ and $(pcl(V))^C$ is a pre-open set containing F and V ∩$(pcl(V))^C$ = φ. Therefore, (X,T) is gp-regular.

(ii) ⇒ (iii):

Let x ∈ X and F be a gp-closed set such that x ∉ F. Then x ∈ F^C and F^C is a gp-open set Now ,By hypothesis, there exists a pre-open set V of x such that pcl(V) ⊆ F^C. Clearly, pcl(V) ∩ F = φ.

(iii) ⇒ (ii):

Let x ∈ X and W be a gp-open set containing x . Since , W^C is gp-closed set and and x ∉ W^C, hence, by hypothesis, there exists a pre-open set U containing x such that pcl(U) ∩ W^C = φ.

Therefore, pcl(V) ⊆ W.

Hereditary property: The following lemmas are helpful in analyzing the hereditary property of gp-regular spaces:

Lemma (2.5): If X_0 ∈ αO(X,T) and A ∈ PO(X,T), then X_0 ∩ A ∈ PO(X_0 , T_{X_0}),[5]

Lemma (2.6)[10]: Suppose B ⊆ A ⊆ X and (X,T) is a space. If A is open & gp-closed in (X,T) and B is a gp-closed in (A, T_A), then B is also gp-closed in (X,T).

Theorem (2.7): If (X,T) is a gp-regular space & Y is an open and gp-closed subset of (X,T) , then the subspace (Y, T_Y) is a gp-regular space.

i.e. gp-regularity is a hereditary property with respect to an open & gp-closed subspace.

Proof: Let F be any gp-closed subset of (Y, T_Y) and y ∉ F so that y∈ F^C. Since, Y is an open & gp-closed set in (X,T), hence, in view of Lemma (2.6), F is gp-closed in (X,T).

Since, (X,T) is gp-regular, then there exist disjoint pre-open sets U & V of (X,T) such that $y \in U$ & $F \subseteq V$.
As Y is also open so Y is α -open and consequently by lemma (2.5), we get $U \cap Y$ & $V \cap Y$ as disjoint pre-open sets of the subspace (Y, T_Y) such that $y \in U \cap Y$ & $F \subseteq U \cap Y$. Hence, (Y, T_Y) is a gp-regular space.
Hence , the theorem.

Preservation theorem: The gp-regularity of a space is preserved under a bijective, gp irresolute and M-pre –open mapping as established in the following theorem.

Theorem (2.8): If f : (X,T)→(Y,σ) be a bijective, gp-irresolute and M – pre-open mapping from a gp-regular (X,T), then (Y,σ) is also gp-regular.

Proof: Let f : (X,T)→(Y,σ) be a bijective, gp-irresolute and M – pre-open mapping from a gp-regular (X,T) to another space (Y,σ) .
Let $y \in Y$ and F be any gp-closed subset of (Y,σ) with $y \notin F$. Recall that a map f : (X,T)→(Y,σ) is known to be gp-irresolute if $f^{-1}(S)$ is gp-closed in X for every gp-closed set S in Y[2].
Hence, $f^{-1}(F)$ is a gp- closed in (Y,σ). Since, f is bijective, let f(x) = y, then $x \notin f^{-1}(y)$. By hypothesis, there exist pre-open sets U & V such that $x \in U$ and $f^{-1}(F) \in V$ with $U \cap V = \varphi$. Since, f is M-pre-open and bijective, we have $y \in f(U)$ and $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \varphi$. Hence, (Y,σ) is a gp-regular space. Hence, the theorem.

III. g -PRE NORMAL SPACES

The weak form of normality called gp-normality in topological spaces is being introduced and studied in this section.

Definition (3.1): A topological space (X,T) is said to be g-pre-normal (in short gp-normal) space iff for any pair of disjoint gp-closed sets A & B, there exist disjoint pre-open sets U & V such that $A \subseteq U$ & $B \subseteq V$.
Transformation of gp-normal space into a gp- regular space occurs only when it is a pre-R_0 space as described through the following theorem(3.2).

Theorem (3.2): Every gp-normal , pre- R_0 space is gp-regular.

Proof: Let (X,T) be gp-normal as well as pre-R_0 space. Let A be any gp-closed set in (X,T) and $x \in X$ with $x \notin A$.
Now, as (X,T) is pre-R , so{x} is gp-closed in (X,T). Since,(X,T) is gp-normal , hence, there exist disjoint pre-open sets U & V in the manner that $x \in \{X\} \subseteq U$ and $A \subseteq V$. Consequently, (X,T) is gp-regular.

Characterization criteria: The following theorems are enunciated to characterize a gp-normal space.

Theorem (3.3): A topological space (X,T) is gp- normal iff every pair of disjoint gp-closed sets A and B there exist a pair of pre-open sets U & V such that $A \subseteq U$ & $B \subseteq V$. and $pcl(U) \cap pcl(V) = \varphi$.

Proof: Let (X,T) be a gp-normal space, and A & B are any two disjoint gp-closed sets. Then the gp-normality provides that there exist a pair of disjoint pre-open sets U & V in the manner that $A \subseteq U$ and $B \subseteq V$.
Now, $U \subseteq pcl(V)$ & $V \subseteq pcl(V)$ & $U \subseteq V^C$, $V \subseteq U^C$ provide that $pcl(U) \cap V = \varphi$ & $pcl(V) \cap U = \varphi \Rightarrow pcl(U) \cap pcl(V) = \varphi$.
Conversely, Let A & B are any pair of disjoint gp-closed sets. By the hypothesis, there exist pre-open sets U & V in the manner that $A \subseteq U$ and $B \subseteq V$ & $pcl(U) \cap pcl(V) = \varphi$.
Of course, $pcl(U) \cap pcl(V) = \varphi \Rightarrow U \cap V \subseteq pcl(U) \cap pcl(V) = \varphi \Rightarrow U \cap V = \varphi$.
Hence, (X,T) is gp-normal.

Theorem(3.4) For a space (X,T) the following are equivalent:

(i) (X,T) is gp-normal.
(ii) for every gp- closed set F and every open set G containing F, there exists a pre open set V such that $F \subseteq U \subseteq pcl(U) \subseteq G$.

Proof: (i) ⇒ (ii):
Let (X,T) is gp-normal space. Let F be a gp-closed set and G be a gp-open set such that $F \subseteq G$, then, $F \cap G^C = \varphi$.
Now, F & G^C are disjoint gp-closed sets in a gp-normal space (X,T). So there exist pre-open sets U & V in the manner that $F \subseteq U$ and $G^C \subseteq V$ where $U \cap V = \varphi$.
Obviously, $V^C \subseteq G$ & $U \cap V = \varphi \Rightarrow U \subseteq V^C$. As V^C is pre-closed so $pcl(U) \subseteq V^C$.
Therefore, $F \subseteq U \subseteq pcl(U) \subseteq G$ holdsgood.
(ii) ⇒ (i):
Given that for every gp- closed set F and every open set G containing F, there exists a pre open set V such that $F \subseteq U \subseteq pcl(U) \subseteq G$ where F,G & U are subsets of a space (X,T).
Now, $pcl(U) \subseteq G \Rightarrow G^C \subseteq \{pcl(V)\}^C$, and $F \cap G^C = \varphi$; also, $U \cap \{pcl(V)\}^C = \varphi$. Thus, F & G^C are disjoint gp-closed sets in (X,T) and U & $\{pcl(V)\}^C$ are disjoint pair of pre-open sets.
Obviously, $F \subseteq U$ & $G^C \subseteq \{pcl(V)\}^C$ i.e. for a pair of disjoint gp-closed set F = A & $G^C = B$, there exist a pair of disjoint pre-open sets U & $V = \{pcl(V)\}^C$ in the manner that $A \subseteq U$ and $B \subseteq V$ where $U \cap V = \varphi$. hence, (X,T) is gp-normal space.

Hereditary criteria: gp-normality is hereditary property with respect to an open and gp-closed subspace.

Theorem (3.5): If (X,T) is a gp-normal space and Y is an open & gp-closed subset of (X,T), then (Y, T_Y) is a gp-normal subspace.

Proof: Let A & B be any two disjoint gp-closed sets of (Y, T_Y) .
Since, Y is an open & gp-closed set in (X,T), hence, in view of Lemma (2.6), A & B are gp-closed in (X,T).
Since, (X,T) is gp-normal, then there exist disjoint pre-open sets U & V of (X,T) such that

A⊆ U & B ⊆ V . As Y is also open so Y is α -open and consequently by lemma (2.5), we get U ∩ Y & V∩ Y as disjoint pre-open sets of the subspace (Y, T_Y) such that A ∈ U ∩ Y & B ⊆ U ∩ Y . Hence, (Y, T_Y) is a gp-normal space.

Preservation criteria: The gp-normality of a space is preserved under a bijective, gp-irresolute amd M—pre-open mapping as expressed in following theorem.

Theorem (3.6): If f : (X,T)→(Y,σ) be a bijective, gp-irresolute and M – pre-open mapping from a gp-normal (X,T), then (Y,σ) is also gp-normal..

Proof: : Let f : (X,T)→(Y,σ) be a bijective, gp-irresolute and M – pre-open mapping from a gp-normal (X,T) to another space (Y,σ) . Let A and B be a pair of disjoint gp-closed sets of (Y,σ). Since, the map f is gp-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint gp-closed sets of (X,T). As (X,T) is gp-normal , there exist pre-open sets U & V such that and $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since, f is M-pre-open and bijective, we have f(U) and f(V) are pre-open sets in (Y,σ) such that A⊆ f(U) & B⊆ f(V) and f(U) ∩ f(V) = φ. Hence, (Y,σ) is a gp- normal space. Hence, the theorem.

IV. CONCLUSION

The generalized pre closed sets are used to introduce the concepts gp-regular & gp-normal space.

Also, the characterization , the preservation & hereditary nature of gp-regular as well as gp-normal spaces have been framed and analyzed.

Transformation of a strongly regulars pace into a gp-regular space under the criteria of being $pre^{*}—T_{1/2}$ has been discussed.

Transformation of a gp-normal space into a gp-regular space under the criteria of being pre-R_0 has also been analyzed.

Of course, the entire content will be a successful tool for the researchers for finding the path to obtain the results in the context of $g^{*}\mathcal{K}$ - regular / normal spaces where $\mathcal{K}$ = p,b & β.

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