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Variational Iteration Method for solving two-factor Commodity Price Model Equation

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Abstract: In this work a two-factor model of stochastic behavior of commodity prices given by Eduardo S. Schwartz is solved using Variational Iteration Method (VIM). Numerical example is studied to demonstrate the accuracy of the present method.

Key words: Commodity models, prices on commodities, series solution, variational iteration method, stochastic differential equation

I. INTRODUCTION

Commodities are generally categorized, on the basis of their nature, as storable and non-storable. Storable goods are durable in nature and can be stores for longer period of time without incurring much on their storage. Non-storable goods need huge costs of maintenance for their storage. However, most commodities do fall between durable and non-durable category. Therefore, this key characteristic must be considered in modeling the price dynamics of different commodities.
The variational iteration method was first proposed by He ([1], [2]) and applied successfully to linear and non-linear type ordinary and partial differential equations ([3]-[9]) fractional order ordinary and partial differential equations [10].The highest stuff of this method is in its flexibility and ability to solve nonlinear equations accurately and conveniently ([11]-[24]).
In this paper a two-factor model of commodity prices given by Eduardo S. Schwartz [25] is considered and solved using the Variational iteration method (VIM). The present paper is organized as sections covering: Introduction in Section 1 (given above), the two-factor commodity price model and its solution discussed in section 2. In section 3, a basic idea of VIM is introduced to solve the mentioned two factor commodity price model and trace its convergence, and finally, a numerical example is shown in section 4.

II. THE MODEL EQUATION AND ITS SOLUTION

A. *The Two-Factor Model*

$$\frac{\sigma_1^2}{2}x^2u_{xx}+\sigma_1\sigma_2\rho x u_{xy}+\frac{\sigma_2^2}{2}u_{yy}+(r-y)xu_x+[k(\alpha-y)-\lambda]u_y-u_t=0 \qquad (2.1)$$

With terminal boundary condition $u(x,y,0)=x$ (2.2)

B. *The Solution in the Literature*

The closed form solution of the above equation (2.1) found to be

$$u(x,y,t)=xexp\left[-y\frac{1-e^{-kt}}{k}+A(t)\right] \qquad (2.3)$$

Where

$$A(t)=\left(r-\left(\alpha-\frac{\lambda}{k}\right)+\frac{\sigma_2^2}{2k^2}-\frac{\sigma_1\sigma_2\rho}{k}\right)t+\frac{\sigma_2^2}{4}\frac{(1-e^{-2kt})}{k^3}+\left(\alpha k-\lambda+\sigma_1\sigma_2\rho-\frac{\sigma_2^2}{k}\right)\frac{(1-e^{-kt})}{k^2} \qquad (2.4)$$

III. VARIATIONAL ITERATION METHOD (VIM)

A. ***Basic Ideas of He's VIM***

Consider the following differential equation

$$Lu+Nu=g(x,t) \qquad (3.1)$$

Where L is linear operator, N is nonlinear operator and $g(x,t)$ is a known real function. According to VIM, a correction functional, $u(x,t)$ is as follows:

$$u_{n+1}=u_n(x,t)+\int_0^t\lambda\{Lu_n(x,\xi)+N\widetilde{u_n}(x,\xi)-g(x,\xi)\}d\xi \qquad (3.2)$$

Where λ is the general Lagrange multiplier, u_0 is an initial approximation, $\widetilde{u_n}(x,t)$ is the restricted variation, i.e. $\delta\widetilde{u_n}=0$. The optimal value of the general Lagrange multipliers λ can be identified by using the stationary conditions of the variational theory.
For sufficiently large values of n we can consider u_n as an approximation of the exact solution.

B. *VIM for the Two-Factor Model Equation*

To obtain the approximate solution to equation (2.1) along with (2.2), according to VIM, it can be written as follows

$$u_{n+1} = u_n(x,y,t) + \int_0^t \phi \left[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho x \frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - (r-y)x\frac{\partial u_n}{\partial x} - \{k(\alpha - y) - \lambda\}\frac{\partial u_n}{\partial y}\right] d\xi \tag{3.3}$$

$$\delta u_{n+1} = \delta u_n(x,y,t) + \delta \int_0^t \phi \left[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 \widetilde{u_n}}{\partial x^2} - \sigma_1\sigma_2\rho x \frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial y^2} - (r-y)x\frac{\partial \widetilde{u_n}}{\partial x} - \{k(\alpha - y) - \lambda\}\frac{\partial \widetilde{u_n}}{\partial y}\right] d\xi \tag{3.4}$$

$$\delta u_{n+1} = \delta u_n(x,y,t) + \delta \int_0^t \phi \left[\frac{\partial u_n}{\partial \xi}\right] d\xi \tag{3.5}$$

$$\delta u_{n+1} = \delta u_n(x,y,t)(1+\phi) - \delta \int_0^t \phi' \delta u_n(x,y,\xi) d\xi \tag{3.6}$$

This yields the stationary conditions

$$1 + \phi = 0,\ \phi' = 0 \Rightarrow \phi = -1 \tag{3.7}$$

Substituting the value of $\phi = -1$ into the functional (3.3) give the iteration formulas

$$u_{n+1} = u_n(x,y,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho x \frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - (r-y)x\frac{\partial u_n}{\partial x} - \{k(\alpha - y) - \lambda\}\frac{\partial u_n}{\partial y}\right] d\xi \tag{3.8}$$

C. Convergence of VIM for Two-Factor Model

Let the terms, $F_1 \equiv x^2 u_{xx}(x,y,t),\ F_2 \equiv x u_{xy}(x,y,t), F_3 \equiv u_{yy}(x,y,t), F_4 \equiv u_y(x,y,t)$, $F_5 \equiv x u_x(x,y,t), F_6 \equiv y\, u_y(x,y,t), F_7 \equiv xy\, u_x(x,y,t)$ are Lipschitz continuous with $|F_1(u) - F_1(u^*)| \le L_1|u - u^*|, |F_2(u) - F_2(u^*)| \le L_2|u - u^*|, |F_3(u) - F_3(u^*)| \le L_3|u - u^*|, |F_4(u) - F_4(u^*)| \le L_4|u - u^*|,\ |F_5(u) - F_5(u^*)| \le L_5|u - u^*|,\ |F_6(u) - F_6(u^*)| \le L_6|u - u^*|$ and $|F_7(u) - F_7(u^*)| \le L_7|u - u^*|$ for $x, y > 0$, and $J = [0,\ T]\ (T \in \mathbb{R})$.

Let $\beta_1 = \{|a|L_1 + |b|L_2 + |c|L_3 + |d|L_4 + |r|L_5 + |k|L_6 + L_7\}$, and $\beta_2 = \{1 - T(1 - \beta_1)\}$ where $a = \frac{\sigma_1^2}{2},\ b = \sigma_1\sigma_2\rho$, $c = \frac{\sigma_2^2}{2}$, and $d = (k\alpha - \lambda)$

Theorem: The solution $u_n(x,y,t)$ obtained from (3.8) converges to the solution of problem (2.1) when $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$.

Proof: From (3.8)

$$u_{n+1} = u_n(x,y,t) - \int_0^t \left[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2} x^2 \frac{\partial^2 u_n}{\partial x^2} - \sigma_1\sigma_2\rho x \frac{\partial^2 u_n}{\partial x\,\partial y} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - (r-y)x\frac{\partial u_n}{\partial x} - \{k(\alpha - y) - \lambda\}\frac{\partial u_n}{\partial y}\right] d\xi$$

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \xi} - a\,F_1(u_n) - bF_2(u_n) - cF_3(u_n) - dF_4(u_n) - rF_5(u_n) + kF_6(u_n) + F_7(u_n)\right] d\xi \tag{3.9}$$

$$u = u - \int_0^t \left[\frac{\partial u}{\partial \xi} - a\,F_1(u) - bF_2(u) - cF_3(u) - dF_4(u) - rF_5(u) + kF_6(u) + F_7(u)\right] d\xi \tag{3.10}$$

Let $e_{n+1}(x,y,t) = u_{n+1}(x,y,t) - u_n(x,y,t),\ e_n(x,y,t) = u_n(x,y,t) - u(x,y,t)$
$|e_n(x,y,t^*)| = \max_t |e_n(x,y,t)|$. Since e_n is a decreasing function with respect to 't' from (3.9), (3.10) and mean value theorem we obtained,

$$e_{n+1} = e_n + \int_0^t \left[\frac{\partial(-e_n)}{\partial \xi} + a\{F_1(u_n) - F_1(u)\} + b\{F_2(u_n) - F_2(u)\} + c\{F_3(u_n) - F_3(u)\} + d\{F_4(u_n) - F_4(u)\} +\right.$$

rF5un−F5u−k F6un−F6u−{F7un−F7(u)}dt

$$e_{n+1} \le e_n + \int_0^t (-e_n)dt + \{|a|L_1 + |b|L_2 + |c|L_3 + |d|L_4 + |r|L_5 + |k|L_6 + L_7\}\int_0^t |e_n|dt$$

$$e_{n+1}(x,y,t) \le e_n(x,y,t) - Te_n(x,y,\omega) + \{|a|L_1 + |b|L_2 + |c|L_3 + |d|L_4 + |r|L_5 + |k|L_6 + L_7\}\int_0^t |e_n|dt$$

$$e_{n+1}(x,y,t) \le e_n(x,y,t) - Te_n(x,y,\ \omega) + \{|a|L_1 + |b|L_2 + |c|L_3 + |d|L_4 + |r|L_5 + |k|L_6 + L_7\}\,T\,|e_n(x,y,t)|$$

$$e_{n+1}(x,y,t) \le \{1 - T(1 - \beta_1)\}\,|e_n(x,y,t^*)|$$

Where $0 \le \omega \le t$, hence $e_{n+1}(x,y,t) \le \beta_2|e_n(x,y,t^*)|$, therefore,

$\|e_{n+1}\| = \max_{\forall t \in J}|e_{n+1}| \le \beta_2 \max_{\forall t \in J}|e_n| \le \beta_2\|e_n\|$

Since, $0 < \beta_2 < 1$, then $\|e_n\| \to 0$.

IV. Numerical Example

Example-1: Consider $\sigma_1 = \sigma_2 = \rho = \lambda = \alpha = r = 1$ and $k = 2$ in the equation (2.3),

We obtain the following approximant using VIM from (3.8)

$$u_4 = \left(\frac{t^2*(y^2-1)}{2} - t*(y-1) + \frac{t^4*(y^4+8*y^3-2*y^2-20*y+7)}{24} - \frac{t^3*y*(y^2+3*y-5)}{6} + 1\right)*x$$

TABLE I

x	y	t	Exact	% error for 4 iterations	% error for 10 iterations	% error for 15 iterations
0.1	0.1	0.1	0.108514650313817	0.000225525169758	0.000000000031435	0.000000000000000
0.1	0.3	0.5	0.126080183826514	0.256160171527562	0.001339926359289	0.000008736883567
0.1	0.4	0.3	0.114831499317686	0.015060921799838	0.000005108907463	0.000000002269532
0.1	0.7	0.5	0.111107030154028	0.241625075679643	0.000445141946416	0.000007741123078

0.2	0.1	0.3	0.245744226581466	0.042552441613084	0.000004684219211	0.000000002477561
0.2	0.3	0.4	0.243997769396631	0.094825431099049	0.000125803582047	0.000000269613210
0.2	0.6	0.5	0.229349532084243	0.113886759888393	0.000215090056100	0.000002499662425
0.2	0.7	0.1	0.205542247404230	0.000086269805227	0.000000000007562	0.000000000000000
0.2	0.7	0.2	0.210348953560144	0.002709256871012	0.000000016932015	0.000000000003800
0.2	0.7	0.3	0.214634108300020	0.020048793903504	0.000001544633820	0.000000002383083
0.2	0.7	0.4	0.218552197520032	0.081928339650712	0.000037690098535	0.000000227652003
0.2	0.7	0.5	0.222214060308057	0.241625075679643	0.000445141946416	0.000007741123078
0.3	0.2	0.1	0.322606726418853	0.000187714978442	0.000000000039249	0.000000000000000
0.3	0.4	0.2	0.331518938090176	0.002357002549765	0.000000066112834	0.000000000003901
0.4	0.6	0.5	0.458699064168487	0.113886759888393	0.000215090056100	0.000002499662425
0.5	0.4	0.3	0.574157496588431	0.015060921799838	0.000005108907468	0.000000002269532
0.5	0.5	0.5	0.591785323961274	0.015703540337538	0.000765528153742	0.000002535499741
0.6	0.5	0.4	0.692774305401533	0.009482828383622	0.000073775609204	0.000000083708376
0.7	0.6	0.4	0.786286631087068	0.036546013761413	0.000023169953887	0.000000068622000
0.7	0.9	0.5	0.730107705857810	0.459438866242626	0.001751471797482	0.000014217944413
0.7	1	0.5	0.707392756839248	0.529680784781616	0.002171834296368	0.000013049826883
0.8	0.6	0.3	0.878124634099885	0.008322520180750	0.000001224325800	0.000000000658036
0.8	1	0.2	0.800799179414821	0.006557542702893	0.000000108880576	0.000000000007029
0.9	0.8	0.5	0.968852635862574	0.359846867678711	0.001135102681645	0.000012060571743
1	0.3	0.5	1.260801838265140	0.256160171527575	0.001339926359272	0.000008736883550
1	0.7	0.4	1.092760987600160	0.081928339650704	0.000037690098543	0.000000227652003
1	1	0.5	1.010561081198930	0.529680784781603	0.002171834296371	0.000013049826893

V. CONCLUSION

The derived results of this study are aimed to prove the effectiveness of the Variational Iteration Method (VIM) used to find solution of Two Factor Model of Stochastic Behavior proposed by Eduardo S. Schwartz on commodity prices. The study applied MATLAB package to calculate the approximate series. The numerical results given in table: 1, depict improvement in solutions obtained by the VIM over large number of iterations in comparison to the exact solution.

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