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Approximate Solution of three-factor Commodity Price Model using Variational Iteration Method

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Abstract: The present work is aimed to solve the three-factor model of stochastic behavior of commodity prices given by Eduardo S. Schwartz using Variational Iteration Method (VIM). The accuracy of the method is proved using numerical example.

Key words: Commodity price models, prices on commodities, variational iteration method, approximate solution, series solution

I. INTRODUCTION

Commodities traded are generally classified, basing on their nature, as storable and non-storable. Storable goods are durables in nature and give a chance longer storage period with relatively lower storage charges. On the other hand, non-storable goods require huge costs on storage. This crucial characteristic, therefore, must be considered in modeling the price dynamics of different commodities. Variational Iteration Method (VIM) was firstly proposed by He ([1], [2]), and was successfully implemented to linear and non-linear type of ordinary and partial differential equations ([3]-[9]), fractional order ordinary and partial differential equation ([10]-[12]) and integral equations ([13]-[15]). The convenience of the method is in its flexibility and ability to solve nonlinear equations accurately ([16]-[29]).

The paper, considers three-factor model of commodity prices given by Eduardo S. Schwartz [30] applies VIM. The paper is organized into the sections namely: 1. Introduction; 2. Two factor model and discussion of its results; 3. Basic idea of VIM to solve the three factor commodity price model and its convergence; 4. A numerical example to prove the output.

II. THE MODEL EQUATIONS AND THEIR SOLUTIONS

A. The three-factor model equation

$$\frac{\sigma_1^2}{2}x^2u_{xx}+\frac{\sigma_2^2}{2}u_{yy}+\frac{\sigma_3^2}{2}u_{zz}+\sigma_1\sigma_2\rho_1xu_{xy}+\sigma_2\sigma_3\rho_2u_{yz}+\sigma_1\sigma_3\rho_3xu_{xz}+(z-y)xu_x+k(\hat{\alpha}-y)u_y+a(m^*-z)u_z-u_t=0 \quad (2.1)$$

With terminal boundary condition $u(x,y,z,0)=x$ (2.2)

B. The solution in the literature

The closed form solution of the above equation found to be

$$u(x,y,z,t)=x\exp\left[-y\frac{1-e^{-kt}}{k}+z\frac{1-e^{-at}}{a}+C(t)\right] \quad (2.3)$$

Where

$$C(t)=\left[\begin{array}{c}\left\{\frac{(k\hat{\alpha}+\sigma_1\sigma_2\rho_1)(1-kt-e^{-kt})}{k^2}\right\}-\frac{\sigma_2^2\{4(1-e^{-kt})-(1-e^{-2kt})-2kt\}}{4k^3}\\ -\frac{\{(am^*+\sigma_1\sigma_3\rho_3)(1-at-e^{-at})\}}{a^2}-\frac{\sigma_3^2\{4(1-e^{-at})-(1-e^{-2at})-2at\}}{4a^3}\\ +\sigma_2\sigma_3\rho_2\left\{\frac{\left((1-e^{-kt})+(1-e^{-at})-(1-e^{-(k+a)t})\right)}{ka(k+a)}+\frac{k^2(1-e^{-at})+a^2(1-e^{-at})-ka^2t-ak^2t}{k^2a^2(k+a)}\right\}\end{array}\right] \quad (2.4)$$

III. VARIATIONAL ITERATION METHOD (VIM)

A. Basic Idea of He's VIM

Consider the following differential equation

$$Lu+Nu=g(x,t) \quad (3.1)$$

Where L is linear operator, N is nonlinear operator and $g(x,t)$ is a known real function. According to VIM, we can construct a correction functional, $u(x,t)$ as follows:

$$u_{n+1}=u_n(x,t)+\int_0^t\lambda\{Lu_n(x,\xi)+N\widetilde{u_n}(x,\xi)-g(x,\xi)\}d\xi \quad (3.2)$$

Where λ is the general Lagrange multiplier, u_0 is an initial approximation, $\widetilde{u_n}(x,t)$ is the restricted variation, i.e. $\delta\widetilde{u_n} = 0$. The optimal value of the general Lagrange multipliers λ can be identified by using the stationary conditions of the variational theory.
For sufficiently large values of n we can consider u_n as an approximation of the exact solution.

B. VIM for the equation of section 2

The approximate solution to the equation (2.1), using VIM, can be written as follows

$$u_{n+1} = u_n(x,y,z,t) + \int_0^t \phi \Big[\frac{\partial u_n(x,y,z,\xi)}{\partial \xi} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 u_n}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 u_n}{\partial y\,\partial z} -$$
σ1σ3ρ3x∂2un ∂x ∂z−z−yx∂un∂x−kα−y∂un∂y−am∗−z∂un∂zdξ (3.3)

$\delta u_{n+1} =$

$$\delta u_n(x,y,z,t) + \delta\int_0^t \phi\Big[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 \widetilde{u_n}}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 \widetilde{u_n}}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 \widetilde{u_n}}{\partial y\,\partial z} - \sigma_1\sigma_3\rho_3 x\frac{\partial^2 \widetilde{u_n}}{\partial x\,\partial z} -$$
z−yx∂un∂x−kα−y∂un∂y−am∗−z∂un∂zdξ (3.4)

$$\delta u_{n+1} = \delta u_n(x,y,z,t) + \delta\int_0^t \phi\left[\frac{\partial u_n}{\partial \xi}\right]d\xi \quad (3.5)$$

$$\delta u_{n+1} = \delta u_n(x,y,z,t)(1+\phi) - \delta\int_0^t \phi'\,\delta u_n(x,y,z,\xi)d\xi \quad (3.6)$$

This yields the stationary conditions

$$1+\phi = 0,\ \phi' = 0 \Longrightarrow \phi = -1 \quad (3.7)$$

Substituting the value of $\phi = -1$ into the functional (3.3) give the iteration formulas

$$u_{n+1} = u_n(x,y,z,t) - \int_0^t \Big[\frac{\partial u_n}{\partial \xi} - \frac{\sigma_1^2}{2}x^2\frac{\partial^2 u_n}{\partial x^2} - \frac{\sigma_2^2}{2}\frac{\partial^2 u_n}{\partial y^2} - \frac{\sigma_3^2}{2}\frac{\partial^2 u_n}{\partial z^2} - \sigma_1\sigma_2\rho_1 x\frac{\partial^2 u_n}{\partial x\,\partial y} - \sigma_2\sigma_3\rho_2\frac{\partial^2 u_n}{\partial y\,\partial z} - \sigma_1\sigma_3\rho_3 x\frac{\partial^2 u_n}{\partial x\,\partial z} -$$
z−yx∂un∂x−kα−y∂un∂y−am∗−z∂un∂zdξ (3.8)

C. Convergence of VIM for three-factor model

Equation (2.1), can be re-written as

$$u_t - c_1x^2u_{xx} - c_2u_{yy} - c_3u_{zz} - c_4xu_{xy} - c_5u_{yz} - c_6xu_{xz} - c_7u_y - c_8u_z + c_9yu_y + c_{10}zu_z - (z-y)xu_x = 0 \quad (3.9)$$

Where $c_1 = \frac{\sigma_1^2}{2}, c_2 = \frac{\sigma_2^2}{2}, c_3 = \frac{\sigma_3^2}{2}, c_4 = \sigma_1\sigma_2\rho_1, c_5 = \sigma_2\sigma_3\rho_2, c_6 = \sigma_1\sigma_3\rho_3, c_7 = k\hat{\alpha},\ c_8 = am^*, c_9 = k, c_{10} = a$
Let $F_1 \equiv x^2u_{xx}, F_2 \equiv u_{yy}, F_3 \equiv u_{zz}, F_4 \equiv xu_{xy}, F_5 \equiv u_{yz}, F_6 \equiv xu_{xz}, F_7 \equiv u_y, F_8 \equiv u_z,$
$F_9 \equiv yu_y, F_{10} \equiv zu_z, F_{11} \equiv (z-y)xu_x$ are Lipschitz continuous with
$|F_i(u) - F_i(u^*)| \le L_i|u - u^*|$ for $x,y,z > 0$, and $J = [0,T]$ $(T \in \mathbb{R})$ where $i = 1$ to 11. (3.10)
Let $\gamma_1 = \sum_{j=1}^{11}|c_j|\,L_j$ and $\gamma_2 = [1 - T(1-\gamma_1)]$, where $c_{11} = 1$. (3.11)

Theorem: The solution $u_n(x,y,z,t)$ obtained from (3.8) converges to the solution of problem (2.1) when $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$.
Proof: consider,

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,t) - \int_0^t \Big[\frac{\partial u_n}{\partial t} - c_1x^2\frac{\partial^2 u_n}{\partial x^2} - c_2\frac{\partial^2 u_n}{\partial y^2} - c_3\frac{\partial^2 u_n}{\partial z^2} - c_4x\frac{\partial^2 u_n}{\partial x\,\partial y} - c_5\frac{\partial^2 u_n}{\partial y\,\partial z} - c_6x\frac{\partial^2 u_n}{\partial x\,\partial z} - c_7\frac{\partial u_n}{\partial y} -$$
c8∂un∂z+c9y∂un∂y+c10z∂un∂z−z−yx∂un∂xdt

$$\Rightarrow u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial t} - \sum_{j=1}^{8} c_j\ F_j(u_n) + c_9F_9(u_n) + c_{10}F_{10}(u_n) - F_{11}(u_n)\right]dt \quad (3.12)$$

Let $u = u - \int_0^t \left[\frac{\partial u}{\partial t} - \sum_{j=1}^{8} c_j\ F_j(u) + c_9F_9(u) + c_{10}F_{10}(u) - F_{11}(u)\right]dt$ (3.13)
Let $e_{n+1}(x,y,z,t) = u_{n+1} - u_n,\ e_n(x,y,z,t) = u_n - u$
$|e_n(x,y,z,t^*)| = \max_t |e_n(x,y,z,t)|$. Since e_n is a decreasing function with respect to 't' then from mean value theorem and (3.10)-(3.13), we obtained,

$$e_{n+1} = e_n + \int_0^t \Big[\frac{\partial(-e_n)}{\partial t} + \sum_{j=1}^{8} c_j\ \{F_j(u_n) - F_j(u)\} - c_9\{F_9(u_n) - F_9(u)\} - c_{10}\{F_{10}(u_n) - F_{10}(u)\} + \{F_{11}(u_n) -$$
F11(u)}dt

$$e_{n+1} \le e_n + \int_0^t(-e_n)dt + \left[\sum_{j=1}^{11}|c_j|\,|L_j|\right]\int_0^t|e_n|dt$$
$$e_{n+1} \le e_n - Te_n(x,y,z,\omega) + \left[\sum_{j=1}^{11}|c_j|\,|L_j|\right]T|e_n|$$
$$e_{n+1}(x,y,z,t) \le [1 - T\{1-\gamma_1\}]|e_n(x,y,z,t^*)| \text{ where } 0 \le \omega \le t$$

Hence, $e_{n+1}(x,y,z,t) \le \gamma_2|e_n(x,y,z,t^*)|$, therefore,
$\|e_{n+1}\| = max_{\forall t\in J}|e_{n+1}| \le \gamma_2\, max_{\forall t\in J}|e_n| \le \gamma_2\|e_n\|$
Since, $0 < \gamma_2 < 1$, then $\|e_n\| \to 0$

IV. NUMERICAL EXAMPLE

Example-1: Consider $\sigma_1 = \sigma_2 = \sigma_3 = \rho_1 = \rho_2 = \rho_3 = a = m^* = 1\ and\ k = \lambda = \alpha = 2$ in the equation (2.1),
We obtain the following approximant for three iterations using VIM from (3.8)

$$u_3 = \left(1 - t(y-z) + \frac{t^2(y^2 - 2yz + 2y + z^2 - z - 1)}{2} - \frac{t^3(y-z+1)(y^2 - 2yz + 5y + z^2 - 2z - 4)}{6}\right)x$$

TABLE I
Errors obtained for the three-factor model calculated against exact and approximate solutions for various iteration approximants

x	Y	z	t	Exact solution	% error for 3 iterations	% error for 5 iterations	% error for 7 iterations
0.1	0.1	0.1	0.2	0.09865	0.04772	0.00136	2.00E-05
0.1	0.1	0.5	0.1	0.10347	4.00E-05	2.00E-05	0
0.1	0.2	0.3	0.3	0.10027	0.17803	0.01233	0.00039
0.1	0.2	0.6	0.4	0.11013	0.12035	0.06419	0.00014
0.1	0.3	0.3	0.4	0.09705	0.67408	0.04317	0.00475
0.1	0.3	0.6	0.2	0.1045	0.01421	0.00097	0
0.1	0.5	0.2	0.1	0.09698	0.00359	0	0
0.1	0.5	0.3	0.4	0.09185	0.76672	0.00602	0.00403
0.1	0.8	0.1	0.3	0.08315	0.09681	0.01641	0.00064
0.2	0.1	0.6	0.3	0.2217	0.02212	0.01319	0.00022
0.2	0.2	0.5	0.4	0.21312	0.26096	0.06711	0.00106
0.2	0.3	0.3	0.3	0.19607	0.21551	0.00833	0.00049
0.2	0.5	0.3	0.1	0.19582	0.00323	0	0
0.2	0.5	0.6	0.4	0.2028	0.45446	0.02483	0.00269
0.2	0.7	0.2	0.4	0.16822	0.62771	0.06719	0.00112
0.4	0.2	0.5	0.4	0.42625	0.26096	0.06711	0.00106
0.5	0.2	0.6	0.4	0.55067	0.12035	0.06419	0.00014
0.5	0.4	0.3	0.2	0.48676	0.04827	0.00039	2.00E-05
0.6	0.2	0.1	0.3	0.57123	0.28238	0.00946	0.0007
0.7	0.5	0.3	0.2	0.67033	0.05048	4.00E-05	2.00E-05
0.7	0.8	0.3	0.2	0.63799	0.03197	0.00104	1.00E-05
0.8	0.3	0.5	0.4	0.82934	0.39081	0.0535	0.00241
1	0.2	0.3	0.4	0.99762	0.56596	0.06481	0.00386
1	0.8	0.6	0.4	0.93362	0.47393	0.0268	0.00028

V. CONCLUSION

The results derived from the study are aimed to prove the effectiveness of the VIM applying the Three Factor Model of Stochastic Behavior as proposed by Eduardo S. Schwartz on the commodity prices. The approximate series are derived using MATLAB package. The numerical results depicted in table: 1, show improved and accurate solutions obtained with reduced and negligible errors by the VIM at larger number of iterations in comparison to the exact solutions.

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