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Dominating Function Theory from Newton to Leibnitz's Approach of Indefinite Integration

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Abstract— In the paper we have summarized the d-function theory, its properties and applications in the indefinite integrals of some indefinite nonintegrable functions discussed by Yadav & Sen and also to some classical nonelementary functions. The integrals not coming under any of the dominating functions form have been left in series and those which are not d-able have been left with a comment that they are not integrable.

Keywords— Nonelementary functions, Existence theorem on indefinite integrability based on d-functions, Dominating, Sequential and Dominating sequential functions, etc.

2010 AMS Subject Classification-- 30E20, 30D10, 30K05

I. INTRODUCTION

The first reported study of indefinite nonintegrable functions (elliptic integrals) was due to John Wallis (1655). Such integrals cannot be evaluated in terms of elementary functions was proved by Joseph Liouville in 1833. Liouville also created a framework for constructive integration by finding out when indefinite integrals of elementary functions are again elementary functions, which laid the foundation of modern integral calculus, Galois differential equations, Symbolic integration, Computer algebra, etc. But no attempts were made to make them integrable or to overcome the problems of their notations. Yadav & Sen [1-7] introduced six standard forms of indefinite nonintegrable functions and proved them by the help of strong Liouville's theorem and its special cases. They [8-10] applied Newton's approach to find their integrals without any mathematical notations. To give mathematical symbols of integrals in series discussed in [8-10], they [11] introduced a standard dominating function. Then they [12-13] introduced different d-functions based on trigonometric, hyperbolic, exponential and logarithmic functions. Applying Newton's approach of integration in series they [12, 15] established a new existence theorem on indefinite integrability and found the indefinite integrals of many classical nonelementary functions. The complete work of d-function theory and its applications has been presented in detail in the Ph. D. thesis submitted by Yadav [16] under the supervision of Dr. Sen.

II. PRELIMINARY IDEAS

Newton and Leibnitz's Approach of Integration: The two principal inventors of calculus Newton and Leibnitz had very distinct approaches to integration. Newton allowed for infinite series solutions and evaluated integrals by expanding functions in power series and integrating term-by-term. In contrast, Leibnitz favoured solutions in finite terms and searched for closed form expressions of integrals. Well into the 18^{th} century, mathematicians expressed different preferences finite vs. infinite series for representations of indefinite integrals. In dominating function theory we have followed both of the approaches.

Dominating Function Theory: Yadav & Sen [11-15] have developed this theory in a series of papers which has been well explained and presented in Yadav's Ph. D. thesis [16]. They have propounded the following terms and properties:

Dominating Function: An infinite series of the form

$$\sum_{n=1}^{\infty}\frac{B_n}{(ax+b)^n}+C_0+\sum_{n=1}^{\infty}A_n(ax+b)^n$$

has been called a dominating function and those functions which can be written in this form have been called dominatable functions by Yadav and Sen [11].

Theorem 1.1: Dominating function is always continuous, differentiable and indefinite integrable, except at the point of discontinuity.

Theorem 1.2: Necessary Condition for Indefinite Integrability:
Every indefinite integrable function is a dominatable function.

Example: e^x, sinx, sinhx, x^2+x+3, log(1+x), etc.

Theorem 1.3: Sufficient Condition for Indefinite Integrability:
If f(x) be a dominatable function, it is indefinite integrable.

$$\text{Example: } e^{x^2}, e^{\frac{1}{x}}, \frac{e^x}{x^n}, \frac{\sin x}{x^n}, \frac{e^{\sin x}}{\cos x}, etc.$$

Theorem 1.4: Necessary and Sufficient Conditions for Indefinite Integrability:
A function f(x) is indefinite integrable if and only if it is dominatable. In other words, if a function f(x) has an indefinite integral, then it is a D-able function and conversely if f(x) is a D-able function, it has an indefinite integral.

III. APPLICATIONS

To find the indefinite integrals we first find the Taylor's (or Laurent's) series expansion of the integrand to show that whether they are d-able function or not to test the conditions of existence theorem of integrability. Thereafter we find their integrals in series and denote them in different forms of dominating functions if possible. In other words we start with Newton's approach of indefinite integration and end with Leibnitz's closed form expressions for the integrals.

Since the functions $\frac{e^{ax^2+b}}{x}$, $\frac{e^{-x}}{x}$, e^{x^2}, e^{-x^2} are d-able functions, they are indefinite integrable and can be expressed in d-functions form as follows:

$$\textbf{1. } \int \frac{e^{ax^2+b}}{x}dx = e^b \int \left(\frac{1}{x} + \frac{ax}{1!} + \frac{a^2x^3}{2!} + \frac{a^3x^5}{3!} + ... \right) dx = e^b \left(\log x + e^{\left\{\frac{a^k}{2k}\right\}x^2} \right) + c, k \neq 0$$

$$\textbf{2. } \int \frac{e^{-x}}{x}dx = \log x - \frac{x}{1!} + \frac{x^2}{2!2} - \frac{x^3}{3!3} + \frac{x^4}{4!4} - .... + c = \log x + e^{\left\{\frac{1}{k}\right\}(-x)} + c, k \neq 0$$

$$\textbf{3. } \int e^{x^2}dx = \int \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + ... \right) dx = \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+1)} \right\} \frac{x^{2k+1}}{k!} + c = xe^{\left\{\frac{1}{(2k+1)}\right\}x^2} + c$$

$$\textbf{4. } \int e^{x^2}dx = \sum_{k=0}^{\infty} \left\{ \frac{1}{(2k+1)} \right\} \frac{(-1)^k x^{2k+1}}{k!} + c = xe^{\left\{\frac{1}{(2k+1)}\right\}(-x^2)} + c$$

Whereas the functions

$$\frac{e^{x^2+x}}{(x+1)}, \frac{e^{x^3+2x^2+x}}{(x^2+2x+1)}, \frac{\tan x}{ax^3+x^2+b}, \frac{\sinh x}{ax^2+b}, \frac{e^{\sin x}}{\cos x}, \frac{e^{\cos x}}{\sin x}, \frac{e^{\tan x}}{\sec^2 x},$$

$$\frac{e^{\sec x}}{\sec x \tan x}, \frac{e^{\sinh x}}{\cosh x}, \frac{e^{\cosh x}}{\sinh x}, \frac{e^{\tanh x}}{\sec h^2 x}, \frac{e^{\sec hx}}{\sec hx \tanh x}, \frac{e^{\sin^2 x}}{\sin 2x}, \frac{e^{\sin x^2}}{2x\cos x^2},$$

$$\frac{(x^2+2)\tan x}{(3x^4+4x)}, \frac{(2x^2+3)\sin(3x^2+2)}{(2x^3+6)}, e^{\sin x}, e^{\cos x}, e^{\tan x}, e^{\sec x}, e^{\sinh x}, e^{\cosh x},$$

$$e^{\tanh x}, e^{\sec hx}, \sin(x^2+3), \cosh(6x^2+b), \tan(x^2+bx+c), \tan(3x^3+bx^2+cx+d)$$

are d-able functions and are indefinite integrable but their integrals cannot be expressed in terms of d-functions form. Such integrals can only be expressed in series as follow:

5. $\int \frac{e^{x^2+x}}{(x+1)}dx = x+\frac{3x^3}{6}-\frac{x^4}{12}+\frac{11x^5}{40}-\frac{7x^6}{60}+\frac{167x^7}{1008}+.......+K$

6. $\int \frac{e^{x^3+2x^2+x}}{(x^2+2x+1)}dx = x-\frac{x^2}{2}+\frac{7x^3}{6}-\frac{17x^4}{24}+\frac{149x^5}{120}-\frac{569x^6}{720}+.......+K$

7. $\int \frac{\tan x}{ax^3+x^2+b}dx = \frac{x^2}{2b}+\frac{x^4(b-3)}{12b^2}-\frac{ax^5}{5b^2}+\frac{x^6(2b^2-5b+15)}{90b^3}-\frac{x^7a(b-6)}{21b^3}+..........+c$

8. $\int \frac{\sinh x}{ax^2+b}dx = \frac{x^2}{2b}+\frac{x^4(b-6a)}{24b^2}+\frac{x^6(120a^2-20ab+b^2)}{720b^3}+.........+c$

9. $\int \frac{e^{\sin x}}{\cos x}dx = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{17x^6}{720}+.........+c$

10. $\int \frac{e^{\cos x}}{\sin x}dx = e\log x-\frac{ex^2}{6}+\frac{37ex^4}{1440}-\frac{347ex^6}{90720}+.........+c$

11. $\int \frac{e^{\tan x}}{\sec^2 x}dx = x+\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{12}+\frac{5x^5}{120}+\frac{17x^6}{720}-\frac{x^7}{1008}+.........+c$

12. $\int \frac{e^{\sec x}}{\sec x\tan x}dx = e\log x-\frac{ex^2}{6}+\frac{37ex^4}{1440}+\frac{283ex^6}{90720}+\frac{1753ex^8}{1209600}+.........+c$

13. $\int \frac{e^{\sinh x}}{\cosh x}dx = x+\frac{x^2}{2}-\frac{x^4}{24}+\frac{x^5}{30}+\frac{17x^6}{720}-\frac{x^7}{210}+.........+c$

14. $\int \frac{e^{\cosh x}}{\sinh x}dx = e\log x+\frac{ex^2}{6}+\frac{37ex^4}{1440}+\frac{347ex^6}{90720}+.........+c$

15. $\int \frac{e^{\tanh x}}{\sec h^2 x}dx = x+\frac{x^2}{2}+\frac{3x^3}{6}+\frac{5x^4}{24}+\frac{13x^5}{120}+\frac{17x^6}{720}+\frac{13x^7}{1680}+.........+c$

16. $\int \frac{e^{\sec hx}}{\sec hx\tanh x}dx = e\log x+\frac{ex^2}{6}+\frac{37ex^4}{1440}-\frac{283ex^6}{90720}+\frac{1753ex^8}{1209600}+.........+c$

17. $\int \frac{e^{\sin^2 x}}{\sin 2x}dx = \frac{1}{2}\log x+\frac{5x^2}{12}+\frac{103x^4}{720}+\frac{163x^6}{4536}+........+c$

18. $\int \frac{e^{\sin x^2}}{2x\cos x^2}dx = \frac{1}{2}\log x+\frac{x^2}{4}+\frac{x^4}{8}+\frac{x^6}{24}+........+c$

19. $\int e^{\sin x}dx = x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^6}{90}-\frac{x^7}{1680}+........+c$

20. $\int e^{\cos x}dx = ex-\frac{ex^3}{6}+\frac{ex^5}{30}-\frac{31ex^7}{5040}+.........+c$

21. $\int e^{\tan x}dx = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{8} + \frac{3x^5}{40} + \frac{37x^6}{720} + \dots\dots\dots + c$

22. $\int e^{\sec x}dx = ex + \frac{ex^3}{6} + \frac{ex^5}{15} + \frac{151ex^7}{5040} + \dots\dots\dots + c$

23. $\int e^{\sinh x}dx = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{5x^5}{120} + \frac{x^6}{60} + \dots\dots\dots\dots + c$

24. $\int e^{\cosh x}dx = ex + \frac{ex^3}{6} + \frac{ex^5}{30} + \frac{31ex^7}{5040} + \dots\dots\dots\dots + c$

25. $\int e^{\tanh x}dx = x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \frac{7x^5}{120} - \frac{x^6}{240} + \dots\dots\dots\dots + c$

26. $\int e^{\sec hx}dx = ex - \frac{ex^3}{6} + \frac{ex^5}{15} - \frac{151ex^7}{5040} + \dots\dots\dots + c$

27. $\int \sin(x^2+3)dx = x\sin 3 + \frac{x^3}{3}\cos 3 - \frac{x^5 \sin 3}{10} - \frac{x^7 \cos 3}{42} + \frac{x^9 \sin 3}{216} + \dots\dots\dots + c$

28. $\int \cosh(6x^2+b)dx = x\cosh b + \frac{6x^3}{3}\sinh b + \frac{18x^5}{5}\cosh b + \frac{36x^7}{7}\sinh b + \dots\dots\dots + c$

29. $\int \tan(x^2+bx+c)dx = x\tan c + \frac{x^2}{2}(b\tan^2 c + b) + \frac{x^3}{3}(b^2\tan^3 c + b^2\tan c + \tan^2 c + 1) + .. + K$

30. $\int \tan(3x^3+bx^2+cx+d)dx = x\tan d + \frac{cx^2}{2}\sec^2 d + \frac{x^3}{3}\sec^2 d(b + c^2\tan d) - \dots\dots\dots + K$

31. $\int \frac{(x^2+2)\tan x}{(3x^4+4x)}dx = \frac{x}{2} + \frac{5x^3}{36} - \frac{3x^4}{32} + \frac{3x^5}{100} - \dots\dots\dots + c$

32. $\int \frac{(2x^2+3)\sin(3x^2+2)}{(2x^3+6)}dx = \frac{\sin 2}{2}x + \frac{x^3}{18}(2\sin 2 + 9\cos 2) - \frac{x^4}{24}\sin 2 + \dots\dots\dots + c$

But the functions

$$e^{\cos echx},\ e^{\coth x},\ e^{\cos ecx},\ e^{\cot x},\ \frac{e^{\cot x}}{\cos ec^2 x},\ \frac{e^{\cos ecx}}{\cos ecx \cot x},\ \frac{e^{\coth x}}{\cos ech^2 x},\ \frac{e^{\cos echx}}{\cos echx \coth x}$$

are not d-able, therefore they are not indefinite integrable.

IV. CONCLUSIONS

From above we find that there are still many functions which are indefinite integrable in series by d-function theory but they cannot be expressed in dominating, sequential and dominating sequential functions form as well as there are lots of functions which are not indefinite integrable in series also. This indicates that a lot of research is still be needed to overcome the problems of indefinite nonintegrable functions and many new functions needed to be introduced.

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