



## A Study on Integral Problems by Using Maple

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**Abstract**—This study uses the mathematical software Maple for the auxiliary tool to evaluate two types of integrals. We can obtain the closed forms of these two types of integrals by using Leibniz differential rule and differentiation with respect to a parameter. At the same time, we provide some integral examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

**Keywords**—integrals, closed forms, Leibniz differential rule, differentiation with respect to a parameter, Maple

### I. INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Mozart, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website ([www.maplesoft.com](http://www.maplesoft.com)) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, [1]-[7] can be adopted as references.

In calculus and engineering mathematics courses, we learnt many methods to solve the integral problems, including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and so on. In this paper, we study the evaluation of the following two types of indefinite integrals which are not easy to obtain their answers by using the methods mentioned above.

$$\int x^n e^{ax} \cos(bx + c) dx \quad (1)$$

$$\int x^n e^{ax} \sin(bx + c) dx \quad (2)$$

where  $n$  is any non-negative integer, and  $a, b, c$  are real numbers,  $b \neq 0$ . We can obtain the closed forms of these two types of indefinite integrals by using Leibniz differential rule and differentiation with respect to a parameter; these are the main results of this paper (i.e., Theorems 1, 2). As for the study of related integral problems can refer to [8]-[16]. On the other hand, we provide some integral examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

### II. MAIN RESULTS

Firstly, we introduce some notations and two formulas used in this study.

*Notations.*

(i) Let  $z = a + ib$  be a complex number, where  $i = \sqrt{-1}$ ,  $a, b$  are real numbers. We denote  $a$  the real part of  $z$  by  $\text{Re}(z)$ , and  $b$  the imaginary part of  $z$  by  $\text{Im}(z)$ .

(ii) the  $k$ -th order derivative of the function  $u(x)$  is denoted by  $\frac{d^k}{dx^k} u(x)$ , where  $k$  is a non-negative integer.

(iii) Suppose  $n, k$  are positive integers, we define  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ , and  $\binom{n}{0} = 1$ .

Euler's formula.

$e^{i\theta} = \cos \theta + i \sin \theta$ , where  $\theta$  is any real number.

DeMoivre's formula.

$(\cos \theta + i \sin \theta)^m = \cos m\theta + i \sin m\theta$ , where  $m$  is any non-negative integer,  $\theta$  is any real number.

Next, we introduce two important theorems used in this paper.

Leibniz differential rule ([17]).

Suppose  $n$  is a non-negative integer,  $f(x)$  and  $g(x)$  are  $n$ -times differentiable functions. Then the  $n$ -th order derivative of the product function  $f(x) \cdot g(x)$ ,

$$(f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$$

Differentiation with respect to a parameter ([18]).

Suppose  $c, d, \lambda, \beta$  are real numbers and the function  $f(a, x)$  is defined on  $[c, d] \times [\lambda, \beta]$ . If  $f(a, x)$  and its partial derivative  $\frac{\partial f}{\partial a}(a, x)$  are continuous functions on  $[c, d] \times [\lambda, \beta]$ . Then  $F(a) = \int_{\lambda}^{\beta} f(a, x) dx$  is differentiable on the

open interval  $(c, d)$ , and its derivative  $\frac{d}{da} F(a) = \int_{\lambda}^{\beta} \frac{\partial f}{\partial a}(a, x) dx$  for all  $a \in (c, d)$ .

Before deriving the major results in this study, we need two lemmas.

Lemma 1. Assume  $a, b, c$  are real numbers,  $a^2 + b^2 \neq 0$ , and  $C$  is a constant. Then the indefinite integral

$$\int e^{ax} \cos(bx + c) dx = \left[ \frac{a}{a^2 + b^2} \cos(bx + c) + \frac{b}{a^2 + b^2} \sin(bx + c) \right] \cdot e^{ax} + C \quad (3)$$

Proof.

$$\begin{aligned} & \int e^{ax} \cos(bx + c) dx \\ &= \int \operatorname{Re}[e^{ax} e^{i(bx+c)}] dx \quad (\text{by Euler's formula}) \\ &= \int \operatorname{Re}[e^{(a+ib)x+ic}] dx \\ &= \operatorname{Re}[\int e^{(a+ib)x+ic} dx] \\ &= \operatorname{Re}\left[\frac{1}{a+ib} e^{(a+ib)x+ic}\right] + C \\ &= \operatorname{Re}\left[\frac{a-ib}{a^2+b^2} e^{ax} [\cos(bx+c) + i \sin(bx+c)]\right] + C \quad (\text{by Euler's formula}) \\ &= \left[ \frac{a}{a^2+b^2} \cos(bx+c) + \frac{b}{a^2+b^2} \sin(bx+c) \right] \cdot e^{ax} + C \quad \blacksquare \end{aligned}$$

Lemma 2. Suppose  $a, b$  are real numbers,  $b \neq 0$ , and  $k$  is any non-negative integer. Then

$$\frac{d^k}{da^k} \left( \frac{a}{a^2 + b^2} \right) = \frac{(-1)^k k!}{(a^2 + b^2)^{(k+1)/2}} \cdot \cos(k+1)\theta \quad (4)$$

$$\frac{d^k}{da^k} \left( \frac{b}{a^2 + b^2} \right) = \frac{(-1)^k k!}{(a^2 + b^2)^{(k+1)/2}} \cdot \sin(k+1)\theta \quad (5)$$

where  $\theta = \cot^{-1} \frac{a}{b}$ .

$$\begin{aligned}
 \text{Proof. } & \frac{d^k}{da^k} \left( \frac{a}{a^2 + b^2} \right) \\
 &= \frac{d^k}{da^k} \left( \frac{1/2}{a + ib} + \frac{1/2}{a - ib} \right) \\
 &= \frac{1}{2} (-1)^k k! \left[ \frac{1}{(a + ib)^{k+1}} + \frac{1}{(a - ib)^{k+1}} \right] \\
 &= \frac{1}{2} (-1)^k k! \cdot \frac{(a + ib)^{k+1} + (a - ib)^{k+1}}{(a^2 + b^2)^{k+1}} \\
 &= \frac{1}{2} (-1)^k k! \cdot \frac{1}{(a^2 + b^2)^{(k+1)/2}} \cdot \left\{ \left[ \frac{a}{(a^2 + b^2)^{1/2}} + i \frac{b}{(a^2 + b^2)^{1/2}} \right]^{k+1} + \left[ \frac{a}{(a^2 + b^2)^{1/2}} - i \frac{b}{(a^2 + b^2)^{1/2}} \right]^{k+1} \right\} \\
 &= \frac{1}{2} (-1)^k k! \cdot \frac{1}{(a^2 + b^2)^{(k+1)/2}} \cdot [(\cos \theta + i \sin \theta)^{k+1} + (\cos \theta - i \sin \theta)^{k+1}] \\
 &= \frac{1}{2} (-1)^k k! \cdot \frac{1}{(a^2 + b^2)^{(k+1)/2}} \cdot \{ [\cos(k+1)\theta + i \sin(k+1)\theta] + [\cos(k+1)\theta - i \sin(k+1)\theta] \} \\
 &\quad \text{(by DeMoivre's formula)} \\
 &= \frac{(-1)^k k!}{(a^2 + b^2)^{(k+1)/2}} \cdot \cos(k+1)\theta.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \frac{d^k}{da^k} \left( \frac{b}{a^2 + b^2} \right) \\
 &= \frac{d^k}{da^k} \left( \frac{-1/2i}{a + ib} + \frac{1/2i}{a - ib} \right) \\
 &= \frac{-1}{2i} \cdot (-1)^k k! \left[ \frac{1}{(a + ib)^{k+1}} - \frac{1}{(a - ib)^{k+1}} \right] \\
 &= \frac{-1}{2i} \cdot (-1)^k k! \cdot \frac{(a - ib)^{k+1} - (a + ib)^{k+1}}{(a^2 + b^2)^{k+1}} \\
 &= \frac{-1}{2i} \cdot (-1)^k k! \cdot \frac{-2i \sin(k+1)\theta}{(a^2 + b^2)^{(k+1)/2}} \\
 &= \frac{(-1)^k k!}{(a^2 + b^2)^{(k+1)/2}} \cdot \sin(k+1)\theta \quad \blacksquare
 \end{aligned}$$

The following is the first major result of this study, we determine the closed form of indefinite integral (1).

*Theorem 1.* Assume  $n$  is any non-negative integer,  $a, b, c$  are real numbers,  $b \neq 0$ , and  $C$  is a constant. Then the indefinite integral

$$\int x^n e^{ax} \cos(bx + c) dx = n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(a^2 + b^2)^{(k+1)/2}} \cdot \cos[(bx + c) - (k+1)\theta] \cdot x^{n-k} e^{ax} + C \quad (6)$$

where  $\theta = \cot^{-1} \frac{a}{b}$ .

*Proof.* Differentiating  $n$ -times with respect to  $a$  on both sides of (3) in Lemma 1, we obtain

$$\begin{aligned} \frac{d^n}{da^n} \int e^{ax} \cos(bx + c) dx &= \frac{d^n}{da^n} \left[ \frac{a}{a^2 + b^2} \cos(bx + c) + \frac{b}{a^2 + b^2} \sin(bx + c) \right] \cdot e^{ax} + C \\ \Rightarrow \int x^n e^{ax} \cos(bx + c) dx &= \sum_{k=0}^n \binom{n}{k} \left[ \frac{a}{a^2 + b^2} \cos(bx + c) + \frac{b}{a^2 + b^2} \sin(bx + c) \right]^{(k)} \cdot (e^{ax})^{(n-k)} + C \end{aligned}$$

(by differentiation with respect to a parameter and Leibniz differential rule)

$$= \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k k!}{(a^2 + b^2)^{(k+1)/2}} [\cos(k+1)\theta \cdot \cos(bx + c) + \sin(k+1)\theta \cdot \sin(bx + c)] \cdot x^{n-k} e^{ax} + C$$

(by Lemma 2)

$$= n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(a^2 + b^2)^{(k+1)/2}} \cdot \cos[(bx + c) - (k+1)\theta] \cdot x^{n-k} e^{ax} + C \quad \blacksquare$$

Using Theorem 1, we immediately obtain the following result.

*Corollary 1.* Let  $n$  be any non-negative integer,  $a, b, c, r$  are real numbers,  $a < 0, b \neq 0$ , then the improper integral

$$\int_r^\infty x^n e^{ax} \cos(bx + c) dx = -n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(a^2 + b^2)^{(k+1)/2}} \cdot \cos[(br + c) - (k+1)\theta] \cdot r^{n-k} e^{ar} \quad (7)$$

where  $\theta = \cot^{-1} \frac{a}{b}$ .

In Theorem 1, if we replace  $c$  by  $c - \frac{\pi}{2}$ , then we can determine the closed form of indefinite integral (2).

*Theorem 2.* If the assumptions are the same as Theorem 1, then the indefinite integral

$$\int x^n e^{ax} \sin(bx + c) dx = n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(a^2 + b^2)^{(k+1)/2}} \cdot \sin[(bx + c) - (k+1)\theta] \cdot x^{n-k} e^{ax} + C \quad (8)$$

where  $\theta = \cot^{-1} \frac{a}{b}$ .

By Theorem 2, we immediately obtain the following result.

*Corollary 2.* Let the assumptions be the same as Corollary 1, then the improper integral

$$\int_r^\infty x^n e^{ax} \sin(bx + c) dx = -n! \sum_{k=0}^n \frac{(-1)^k}{(n-k)!(a^2 + b^2)^{(k+1)/2}} \cdot \sin[(br + c) - (k+1)\theta] \cdot r^{n-k} e^{ar} \quad (9)$$

where  $\theta = \cot^{-1} \frac{a}{b}$ .

### III. EXAMPLES

In the following, aimed at the two types of integrals in this study, we propose some integral examples and use Theorems 1, 2 and Corollaries 1, 2 to determine their closed forms. On the other hand, we use Maple to calculate the approximations of some related definite integrals and their closed forms for verifying our answers.

*Example 1.* In Theorem 1, if we take  $n = 4, a = 2, b = 3, c = \frac{\pi}{6}$ , then the indefinite integral

$$\int x^4 e^{2x} \cos\left(3x + \frac{\pi}{6}\right) dx = 24 \cdot \sum_{k=0}^4 \frac{(-1)^k}{(4-k)! \cdot 13^{(k+1)/2}} \cdot \cos\left[\left(3x + \frac{\pi}{6}\right) - (k+1) \cot^{-1} \frac{2}{3}\right] \cdot x^{4-k} e^{2x} + C$$

Therefore, we can determine the definite integral

$$\int_6^{14} x^4 e^{2x} \cos\left(3x + \frac{\pi}{6}\right) dx$$

$$= 24 \cdot \sum_{k=0}^4 \frac{(-1)^k}{(4-k)! \cdot 13^{(k+1)/2}} \cdot \cos\left[\left(42 + \frac{\pi}{6}\right) - (k+1) \cot^{-1} \frac{2}{3}\right] \cdot 14^{4-k} e^{28}$$

$$- 24 \cdot \sum_{k=0}^4 \frac{(-1)^k}{(4-k)! \cdot 13^{(k+1)/2}} \cdot \cos\left[\left(18 + \frac{\pi}{6}\right) - (k+1) \cot^{-1} \frac{2}{3}\right] \cdot 6^{4-k} e^{12} \quad (11)$$

Next, we employ Maple to verify the correctness of (11).

>evalf(int(x^4\*exp(2\*x)\*cos(3\*x+Pi/6),x=6..14),18);

$$-1.06357088764881117 \cdot 10^{16}$$

>evalf(24\*sum((-1)^k\*cos(42+(1/6)\*Pi-(k+1)\*arccot(2/3))\*14^(4-k)\*exp(28)/((4-k)!\*13^((k+1)/2)),k=0..4)-24\*sum((-1)^k\*cos(18+(1/6)\*Pi-(k+1)\*arccot(2/3))\*6^(4-k)\*exp(12)/((4-k)!\*13^((k+1)/2)),k=0..4),18);

$$-1.06357088764881117 \cdot 10^{16}$$

Example 2. In Corollary 1, if  $n = 6, r = 4, a = -3, b = 5, c = \frac{3\pi}{8}$ , then we obtain the improper integral

$$\int_4^{\infty} x^6 e^{-3x} \cos\left(5x + \frac{3\pi}{8}\right) dx$$

$$= -720 \cdot \sum_{k=0}^6 \frac{(-1)^k}{(6-k)! \cdot 34^{(k+1)/2}} \cdot \cos\left[\left(20 + \frac{3\pi}{8}\right) - (k+1) \cot^{-1} \frac{-3}{5}\right] \cdot 4^{6-k} e^{-12} \quad (12)$$

We also use Maple to verify the correctness of (12).

>evalf(int(x^6\*exp(-3\*x)\*cos(5\*x+3\*Pi/8),x=4..infinity),18);

$$-0.00437051796546954839$$

>evalf(-720\*sum((-1)^k\*cos(20+(3/8)\*Pi-(k+1)\*arccot(-3/5))\*4^(6-k)\*exp(-12)/((6-k)!\*34^((k+1)/2)),k=0..6),18);

$$-0.00437051796546954830$$

Example 3. In Theorem 2, taking  $n = 3, a = 4, b = 7, c = -\frac{3\pi}{4}$ , we have the indefinite integral

$$\int x^3 e^{4x} \sin\left(7x - \frac{3\pi}{4}\right) dx = 6 \cdot \sum_{k=0}^3 \frac{(-1)^k}{(3-k)! \cdot 65^{(k+1)/2}} \cdot \sin\left[\left(7x - \frac{3\pi}{4}\right) - (k+1) \cot^{-1} \frac{4}{7}\right] \cdot x^{3-k} e^{4x} + C \quad (13)$$

Thus, we can evaluate the definite integral

$$\int_2^9 x^3 e^{4x} \sin\left(7x - \frac{3\pi}{4}\right) dx$$

$$= 6 \cdot \sum_{k=0}^3 \frac{(-1)^k}{(3-k)! \cdot 65^{(k+1)/2}} \cdot \sin\left[\left(63 - \frac{3\pi}{4}\right) - (k+1) \cot^{-1} \frac{4}{7}\right] \cdot 9^{3-k} e^{36}$$

$$- 6 \cdot \sum_{k=0}^3 \frac{(-1)^k}{(3-k)! \cdot 65^{(k+1)/2}} \cdot \sin\left[\left(14 - \frac{3\pi}{4}\right) - (k+1) \cot^{-1} \frac{4}{7}\right] \cdot 2^{3-k} e^8 \quad (14)$$

Next, we employ Maple to verify the correctness of (14).

>evalf(int(x^3\*exp(4\*x)\*sin(7\*x-3\*Pi/4),x=2..9),22);

$$2.383247187828160095462 \cdot 10^{16}$$

>evalf(6\*sum((-1)^k\*sin(63-(3/4)\*Pi-(k+1)\*arccot(4/7))\*9^(3-k)\*exp(36)/((3-k)!\*65^((k+1)/2)),k=0..3)-6\*sum((-1)^k\*sin(14-(3/4)\*Pi-(k+1)\*arccot(4/7))\*2^(3-k)\*exp(8)/((3-k)!\*65^((k+1)/2)), k = 0..3), 24);

$$2.38324718782816009546588 \cdot 10^{16}$$

Example 4. In Corollary 2, let  $n = 7, r = 8, a = -5, b = 2, c = -\frac{4\pi}{9}$ , then the improper integral

$$\int_8^{\infty} x^7 e^{-5x} \sin\left(2x - \frac{4\pi}{9}\right) dx$$

$$= -5040 \cdot \sum_{k=0}^7 \frac{(-1)^k}{(7-k)! \cdot 29^{(k+1)/2}} \cdot \sin\left[\left(16 - \frac{4\pi}{9}\right) - (k+1) \cot^{-1} \frac{-5}{2}\right] \cdot 8^{7-k} e^{-40} \quad (15)$$

Using Maple to verify the correctness of (15) as follows:

>evalf(int(x^7\*exp(-5\*x)\*sin(2\*x-4\*Pi/9),x=8..infinity),20);

$$1.1825204528003497246 \cdot 10^{-12}$$

>evalf(-5040\*sum((-1)^k\*sin(16-(4/9)\*Pi-(k+1)\*arccot(-5/2))\*8^(7-k)\*exp(-40)/((7-k)!\*29^((k+1)/2)), k = 0..7), 20);

$$1.1825204528003497245 \cdot 10^{-12}$$

#### IV. CONCLUSIONS

From the above discussion, we know the Leibniz differential rule and the differentiation with respect to a parameter play significant roles in the theoretical inferences of this study. In fact, the applications of these two theorems are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications.

On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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