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Complementary Tree Domination Number of Interval Graphs

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Abstract: Interval graphs have found applications in a wide range of fields such as scheduling and genetics, among others. In this paper, we put forth the findings related to complementary tree domination number of interval graphs. The exact value of complementary tree domination number and minimal complementary tree domination sets of some particular classes of interval graphs are obtained.
Mathematics Subject Classification: 05C69

Key words: Interval graphs, domination number, complementary tree domination number

1. Introduction

Graphs considered in this paper are all undirected, connected and simple graphs. Throughout this paper, for the graph G = (V, E) and for $S \subseteq V$, the subgraph of G induced by the vertices in S is denoted by ‹ S ›. For any vertex $v \in V$ (G), N (v) denotes the open neighborhood of v and is defined as the set of all vertices adjacent to v in G and N {v} denotes the closed neighborhood of v and is defined as N {v} = N (v) $\cup$ {v}. A vertex of degree one is called a support.

A subset S of the vertex set V of the graph G = (V, E) is a dominating set of the graph G if every vertex not in S is adjacent to a vertex in S. The domination number of the graph G denoted by $\gamma(G)$ and is the minimum cardinality of a dominating set of G. A dominating set $S \subseteq V$ of a graph G with vertex set V (G) and edge set E (G) is a complementary tree domination set if the induced subgraph $< V - S >$ is a tree. Complementary tree domination number is the minimum cardinality of a complementary tree dominating set of G. It is denoted by $\gamma_{ctd}(G)$. The notion of dominating set is due to Ore [1]. Cockayne and Hedetnieme [2] contributed to the domination theory in graphs besides many others. For more details about the domination number one can refer to Walikar et al. [3]. The notion of complementary tree dominating set is due to S. Muttamai et al. [4]. Some results pertaining to the bounds of Complementary tree domination number are obtained by them.

Interval graphs are a special class of circular- arc graphs that can be represented with a set of arcs that do not cover the entire circle. The extensive study of interval graphs has been done for several decades by both mathematicians and computer scientists.

Let I = $\{I_1, I_2, I_3, \ldots, I_k\}$ be an interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$, for i = 1, 2, 3, …, k. Here a_i is called the left end point and b_i is called the right end point. Without loss of generality, one can assume that, all end points of the intervals are distinct numbers between 1 and 2k.. The intervals are named in the increasing order of their right end points. The graph G (V, E) is an interval graph if there is one-to-one correspondence between the vertex set V and the interval family I. Two vertices of G are joined by an edge if and only if their corresponding intervals in I intersect. That is if $I_i=[a_i, b_i]$ and $I_j=[a_j, b_j]$, then I_i and I_j will intersect if $a_i < b_j$ or $a_j < b_i$. Interval graphs are rich in combinatorial structures and have found applications in several disciplines such as traffic control, ecology, biology, computer sciences and particularly useful in cyclic scheduling and computers storage allocation problems etc. Having a representation of graph with intervals or arcs can be helpful in combinatorial problems of the graph, such as isomorphism testing and finding maximum independent set and cliques of graphs.

2. Observations

Observation 2.1: Bounds of complementary tree domination number
Let G be a connected interval graph of order $k \geq 2$. Then $\gamma_{ctd}(G) \leq k-1$
Observation 2.2: Relation between domination number and complementary tree domination number
Let G be a connected interval graph of order $k \geq 2$. Then $\gamma(G) \leq \gamma_{ctd}(G)$
For any graph G, every complementary tree dominating set is a dominating set. But every dominating set need not be a complementary tree dominating set. Hence the result follows.

3. Important Results

The following results obtained by S. Muttamai et al.[4] characterize ctd-sets.
Result 3. 1: Every pendant vertex is a member of all ctd-sets.
Result 3. 2: A ctd-set S of G is minimal if and only if for each vertex v in S, one of the following conditions holds.
(i) V is an isolated vertex of S.
(ii) There exists a vertex in V-S for which $N(u) \cap S = \{v\}$

(iii) $N(v) \cap (V - D) = \phi$

(iv) The induced sub graph $< V - S \cup \{v\} >$, either contains a cycle or disconnected.

4. Complementary Tree Domination Number Of Interval Graphs

In this section, the exact values of complementary tree domination number and minimal complementary tree domination sets of some particular classes of interval graphs are obtained.

Theorem 4. 1: Let $I = \{I_1, I_2, \ldots, I_k\}$, $k \geq 2$ be an interval family corresponding to an interval graph G. Suppose that there exists an interval $I_i \in I$ such that every other interval of the family can't but simply dominate any other interval except the interval I_i, then

$$\gamma_{ctd}(G) = k-1$$

Proof: Let v_i be the vertex corresponding to the interval I_i respectively for i = 1, 2, 3, …, k. Let the interval family $I = \{I_1, I_2, \ldots, I_k\}$, $k \geq 2$ satisfy the condition mentioned in the theorem. Then the vertices $v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_k$ are pendant vertices as they are adjacent to one and only one vertex v_i. But every pendant vertex is a member of all ctd-sets. Thus

$$\gamma_{ctd}(G) \geq k-1 \quad \ldots\ldots\ldots\ldots (1)$$

For any connected graph G of order $k \geq 2$,

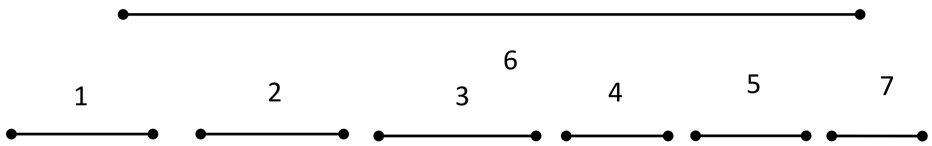
$$\gamma_{ctd}(G) \leq k-1 \quad \ldots\ldots\ldots\ldots (2)$$

From (1) and (2). It follows that

$$\gamma_{ctd}(G) = k-1$$

and minimal complementary tree dominating set is $\{ v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_k \}$.

Illustration 4. 1. 1: Let the interval family I = {1, 2, 3,…., 7} corresponding to the interval graph G be as follows:



Interval family

Clearly the interval family satisfies the conditions mentioned in the theorem 4.1 for k=7. Therefore the complementary tree domination number of the graph G = k-1 = 7-1= 6. Minimal complementary tree dominating set is { 1, 2, 3, 4, 5, 7 }.

Theorem 4. 2: Let $I = \{I_1, I_2, \ldots, I_k\}$, $k \geq 4$ be an interval family analogous to an interval graph G. In a condition, wherein the intervals I_i, I_j are intersecting intervals and any interval of I other than I_i and I_j doesn't dominate any other interval except I_i or I_j, but not both, then

1. $\gamma_{ctd}(G) = k-2$ with minimal ctd-set $I - \{ I_i, I_j \}$, if some of the intervals in $I-\{ I_i, I_j \}$ dominate I_i and some of the intervals in $I-\{ I_i, I_j \}$ dominate I_j and

2 $\gamma_{ctd}(G) = k-1$, if all the intervals in $I-\{ I_i, I_j \}$ dominate I_i or all the intervals in $I-\{ I_i, I_j \}$ dominate I_j .

Proof: Let the interval family $I = \{I_1, I_2, \ldots, I_k\}$, $k \geq 4$ satisfy the conditions mentioned in the hypothesis. Here two cases may arise

Case i: Some of the intervals in $I-\{I_i, I_j\}$ may dominate I_i and the remaining intervals in $I-\{I_i, I_j\}$ may dominate I_j. Let $v_1, v_2, v_3, \ldots, v_k$ be the vertices corresponding to the intervals $I_1, I_2, \ldots, I_k$ respectively. By the conditions for domination between the intervals of the interval family I, it is clear that except the vertices v_i and v_j, the remaining k-2 vertices are pendant vertices. Every pendant vertex is a member of complementary tree dominating set. Therefore

$$\gamma_{ctd}(G) \geq k-2 \quad \ldots\ldots\ldots\ldots (1)$$

Let $S = V - \{v_i, v_j\}$. Then $V - S = \{v_i, v_j\}$. Since, some of the vertices in V are adjacent to v_i and some of the vertices are adjacent to v_j, vertices v_i and v_j are adjacent to at least one vertex in the set S. The set S is a dominating set. As the intervals I_i and I_j are intersecting intervals, the induced subgraph $< V - S >$ consists of two vertices v_i and v_j with an edge between them. Implies subgraph $< V - S >$ is a tree. It follows that S is a ctd-set, wherein the cardinality of S is k-2. Hence

$$\gamma_{ctd}(G) \leq k-2 \quad \ldots\ldots\ldots\ldots (2)$$

From (1) and (2), it is clear that $\gamma_{ctd}(G) = k-2$ with minimal ctd-set $I - \{ I_i, I_j \}$.

Case (ii): All the intervals in $I-\{I_i, I_j\}$ may dominate only the interval I_i. Then all the intervals of I other than I_i will not dominate any other interval except I_i. Therefore, all the vertices of the vertex set V except v_i are pendant vertices. As a result every ctd-set contains all the vertices except v_i. Implies

$$\gamma_{ctd}(G) \geq k-1 \quad \ldots\ldots\ldots\ldots (1)$$

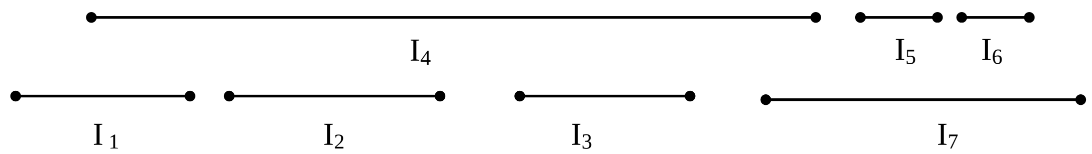
But for every connected graph

$$\gamma_{ctd}(G) \leq k-1 \quad \ldots\ldots\ldots\ldots (2)$$

Hence, $\gamma_{ctd}(G) = k-1$ with minimal ctd-set $I - \{ I_i \}$.

Case (iii): All the intervals in $I-\{I_i, I_j\}$ may dominate only the interval I_j. Then all the intervals of I other than I_j will not dominate any other interval except I_j. By the similar type of argument as in the previous case, it can be proved that $\gamma_{ctd}(G) = k-1$ and the minimal ctd-set in this case as $I - \{I_j \}$. Hence the theorem.

Illustration 4. 2. 1: Let the interval family I = {$I_1, I_2, I_3, \ldots, I_7$} corresponding to the interval graph G be as follows:

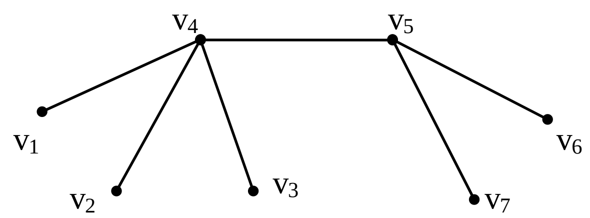


Interval family

Here, the interval I_4 intersects the interval I_5 and any interval of I other than I_4 and I_5 does not dominate any other interval except I_4 or I_5, but not both. Moreover some of the intervals in I-{I_4, I_5} dominate I_4 and some of the intervals in I-{I_4, I_5} dominate I_5. It follows that conditions mentioned in the theorem 4. 2 are satisfied for i = 4 and j = 5 (case i). Hence, $\gamma_{ctd}(G)$ = k-2 = 7-2 = 5

Verification: Let $v_1, v_2, \ldots, v_7$ be the vertices of the interval graph G corresponding to the intervals $I_1, I_2, \ldots, I_7$ of the interval family I respectively.

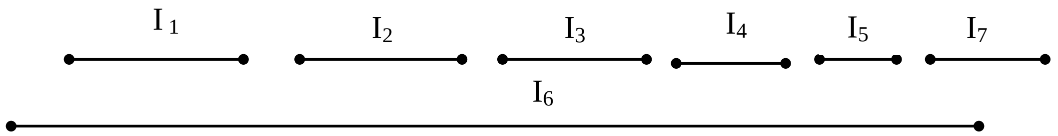
Clearly the interval family satisfies the conditions mentioned in the theorem 4.2 (case i) for k=7. The graph G corresponding to the interval family I will be as follows



Interval Graph

Clearly, $\gamma_{ctd}(G)$ = 5 with minimal ctd-set = { v_1, v_2, v_3, v_6, v_7 }

Illustration 4. 2. 2: Let the interval family I = {$I_1, I_2, I_3, \ldots, I_7$} corresponding to the interval graph G be as follows:

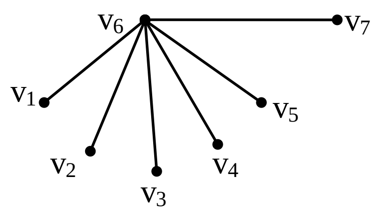


Interval family

Here, the interval I_6 intersects the interval I_7 and any interval of I other than I_6 and I_7 does not dominate any other interval except I_6.. It follows that conditions mentioned in the theorem 4. 2 are satisfied for i = 6 and j = 7 (case ii). Hence, $\gamma_{ctd}(G)$ = k-1 = 7-1 = 6 with the set of arcs { $I_1, I_2, I_3, I_4, I_5, I_7$ } as the minimal ctd-set.

Verification: Let $v_1, v_2, \ldots, v_7$ be the vertices of the interval graph G corresponding to the intervals $I_1, I_2, \ldots, I_7$ of the interval family I respectively.

Clearly the interval family satisfies the conditions mentioned in the theorem 4.2 (case ii) for k=7. The interval graph G corresponding to the interval family I will be as follows



Interval Graph

Clearly, $\gamma_{ctd}(G)$ = 6 with minimal ctd-set { $v_1, v_2, v_3, v_4, v_5, v_7$ }.

Theorem 4.3: In the family of intervals I = { $I_1, I_2, \ldots, I_{2k}$ } corresponding to the Interval graph G, suppose that no other intersections are observed except the following:

(i) I_i intersects no other interval except I_{i+1} which contains only the interval I_i
for i =1, 3, 5, …, 2k-1 and

(ii) I_i intersects no other non contained intervals except I_{i-2} and I_{i+2}
for i=4, 6, …, 2(k-1)

Then, $\gamma_{ctd}(G) = k$ for $k \geq 3$, where $k \in N$ with the set of intervals{ $I_1, I_3, \ldots, I_{2k-1}$ }as the minimal ctd-set.

Proof : Let the interval family I = { $I_1, I_2, \ldots, I_{2k}$ } corresponding to the interval family G satisfy the conditions mentioned in the theorem. By first condition stated in the theorem

I_1 intersects no other interval except $I_{2;}$

I_3 intersects no other interval except I_4;
I_5 intersects no other interval except I_6;
……………………………………………;
……………………………………………;
I_{2k-1} intersects no other interval except I_{2k};
By the first and second conditions stated in the theorem
I_2 intersects no other interval except I_1 and I_4;
I_4 intersects no other interval except I_2, I_3 and I_6;
I_6 intersects no other interval except I_4, I_5 and I_8;
……………………………………………………;
……………………………………………………;
I_{2k-2} intersects no other interval except I_{2k-4}, I_{2k-3} and I_{2k};
I_{2k}intersects no other interval except I_{2k-2}, and I_{2k-1};

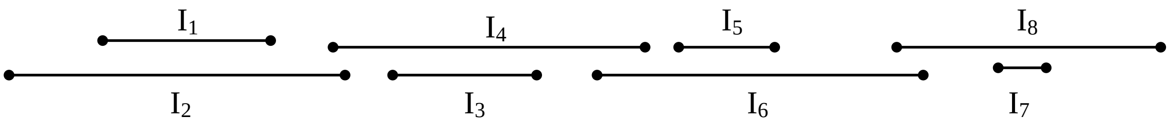
Let $v_1, v_2, v_3, \ldots, v_{2k}$ be the vertices corresponding to the intervals $I_1, I_2, \ldots, I_{2k}$ respectively. Every dominating set of G contains either v_i or v_{i+1} for i =1, 3, 5, …, 2k-1.As a result the set of intervals { $v_1, v_3, \ldots, v_{2k-1}$ } and { $v_2, v_4, \ldots, v_{2k}$} are minimal dominating sets of the graph G. It follows that, $\gamma(G) = K$. By the second condition of the theorem, it is clear that the vertices $v_1, v_3, \ldots, v_{2k-1}$ are the only pendant vertices of the interval graph. Every ctd-set contains each pendant vertex of the graph. It follows that

$$\gamma_{ctd}(G) \geq k$$

Let S = { $v_1, v_3, \ldots, v_{2k-1}$ }. Thus V - S = { $v_2, v_4, \ldots, v_{2k}$}. The set S is a dominating set and the induced subgraph < V – S > is a path graph which is a tree. The set S is a ctd-set. Implies

$$\gamma_{ctd}(G) \leq k$$

As a result, $\gamma_{ctd}(G) = k$ with the set of vertices{ $v_1, v_3, \ldots, v_{2k-1}$ }i.e., the set of arcs { $I_1, I_3, \ldots, I_{2k-1}$ } as the minimal ctd-set.
Illustration 4. 3. 1: Let the interval family I ={ $I_1, I_2, I_3, \ldots, I_8$} corresponding to the interval graph G be as follows:



Interval family I

Clearly the interval family satisfies the conditions mentioned in the theorem 4.3 for k=4. Therefore, $\gamma_{ctd}(G) = k = 4$. The minimal ctd-set is { I_1, I_3, I_5, I_7}

Theorem 4.4: Let I = {$I_1, I_2, I_3, \ldots, I_k$} be the interval family corresponding to an interval graph G. For any three consecutive intervals I_i, I_j and I_k, if I_j doesn't dominate any other interval except I_i and I_k then

$$\gamma_{ctd}(G) = k$$

Proof: Let G be the interval graph, whose interval family I = {$I_1, I_2, I_3, \ldots, I_k$} satisfies the condition mentioned in the theorem. By the hypothesis,
the interval I_1 intersects the interval I_2;
the interval I_2 intersects the intervals I_1 and I_3;
the interval I_3 intersects the intervals I_2 and I_4;
the interval I_4 intersects the intervals I_3 and I_5;
………………………………………...;
…………………………………………;
the interval I_{K-2} intersects the intervals I_{K-3} and I_{K-1};
the interval I_{K-1} intersects the intervals I_{K-2} and I_K;
the interval I_k intersects the interval I_{K-1}.

Let v_1, v_2, v_3, **……,** v_n be the vertices corresponding to the intervals $I_1, I_2, I_3, \ldots, I_k$ respectively. Let S_i = { $v_1, v_2, \ldots, v_{i-1}, v_{i+2}, v_{i+3}, \ldots, v_k$} for i = 2, 3, …., k-2. Then

$$V - S_i = \{ v_i, v_{i+1} \} \text{ for } i = 2, 3, \ldots, k-2.$$

Since I_{i-1}, I_i and I_{i+1} are three consecutive arcs, I_i dominates I_{i-1} and as I_i, I_{i+1} and I_{i+2} are three consecutive arcs, I_{i+1} dominates I_{i+2} for i = 2, 3, …., k-2. Every vertex in V - S_i is adjacent to some vertex in S_i. So the set S_i is a dominating set and also the induced subgraph < V – S_i > consists of only two vertices v_i and v_{i+1} with an edge between them. Impies, the induced subgraph < V – S_i > is a tree. Therefore, the set S_i is a ctd-set for each i = 2, 3,…,k-2. Cardinality of S_i is k-2. It follows that

$$\gamma_{ctd}(G) \leq k - 2.$$

Any set with cardinality k - 2 other than S_i for i = 2, 3, …, k-2 is not a ctd-set. Now we shall show that any subset of S_i is not a dominating set. Here three cases will arise.
Case (i): Let $S_i' = S_i - \{v_1\}$. In this case again two sub cases will arise.
Sub case (i): Let $i \neq 2$. Then v_1 is neither adjacent to v_i nor v_{i+1}. The induced subgraph < V - S_i' > consists of isolated vertex v_1. The induced subgraph < V - S_i' > is a disconnected graph and hence not a tree. Therefore, the set S_i' is not a ctd-set.

Sub case (ii): Let i = 2. Then $S_i' = S_i - \{v_1\} = \{ v_4, v_5, \ldots\ldots, v_k\}$, where i =2. In this case, the induced subgraph < V - S_i' > is a tree with vertex set $\{ v_1, v_2, v_3\}$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_1, v_3\}$. But the set S_i' is not a dominating set since the vertex v_2 in the set V - S_i' is not adjacent to none of the vertices in S_i'. Thus the set S_i' is not a ctd-set. In both the cases, S_i' is not a ctd-set.

Case (ii): Let $S_i' = S_i - \{v_k\}$. When $i \neq k-2$, the induced subgraph < V - S_i' > is not a tree and when i = k-2, the set S_i' is not a dominating set. Hence in both the cases S_i' is not a ctd-set.

Case (iii): Let $S_i' = S_i - \{v_j\}$, where $j \neq 1$, k. With the similar type of argument it can be proved that S_i' is not a ctd-set.

From all the possible three cases, it is clear that any sub set of S_i is not a ctd-set. Implies

$$\gamma_{ctd}(G) \geq k - 2$$

Hence, $\gamma_{ctd}(G) = k - 2$. In this case, the interval family has more than one minimal ctd-set. Minimal ctd-sets are given by $S_i = \{ v_1, v_2, \ldots\ldots, v_{i-1}, v_{i+2}, v_{i+3}, \ldots, v_k\}$ i.e., the family $\{ I_1, I_2, \ldots\ldots, I_{i-1}, I_{i+2}, I_{i+3}, \ldots, I_k\}$, where i = 2, 3, …., k-2.

Illustration 4. 4. 1: Let the interval family $I = \{I_1, I_2, I_3, \ldots., I_8\}$ corresponding to the interval graph G be as follows:

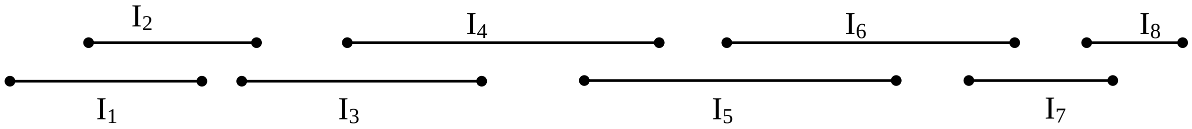


Fig. a: Interval family

Clearly the interval family I satisfy the conditions mentioned in the theorem 4. 4 for k=8. Therefore, $\gamma_{ctd}(G) = k - 2 = 8 - 2 = 6$ with one of the minimal ctd-set as {1, 2, 3, 4, 7, 8}

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