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Computation of Three Species Ecological Model By Homotopy Analysis Method

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Abstract:** in this paper, Homotopy analysis method is proposed to solve a three species ecological model with a Prey (N_1), a predator (N_2) and a competitor (N_3) to both the prey (N_1) and predator (N_2) .The model is characterized by a set of first order non-linear ordinary differential equations. When dealing with nonlinear differential equations, it is difficult to obtain the closed form solutions. Finding accurate and efficient methods for solving non linear problems has long been active research .Many methods such as perturbation method, non perturbation methods like δ-expansion method, Lyapunov's artificial small parameter method, Adomians decomposition method and so on, applied to solve the non linear problems. These non perturbation cannot ensure convergence so we employee the HAM to obtained the series solution of the non-linear system and compare the results with exact solutions using Numerical simulation.

***Key Words:** prey, predator, competitor, embedded parameter, linear operator, HAM and zero order deformation equation.*

1. INTRODUCTION

Ecology relates to the study of living beings (animals and plants) in relation to their habits and habitats. The problem of species distribution is classified as prey- predator, Commensalism, Ammensalism, Mutualism, Competition, Parasitism and so on. The modeling of Eco systems consists of functional units of ecology and system of first order non linear differential equations .Research in the area of theoretical ecology was initiated in 1925 by Lotka [5] and by Volterra [12]. The general concept of modeling have been presented in the treatises of Paul Colinvaux [6], Freedman [2], Kapur [3,4] etc . In this paper a three species Ecological Model with a Prey, Predator and a Competitor to both the prey and predator species is considered for approximating the series solution of the non-linear by the Homotopy analysis method (HAM) and the results are compare with exact solutions. Liao [7, 8, 9, 10]. Proposed the idea of Homotopy analysis method based on Homotopy of topology, which is a powerful analytical method for solving nonlinear problems .Ham ensure the convergence of the series solutions, which depends on auxiliary parameter and auxiliary function. Ham provides great freedom to construct the base function to approximate the series solutions. Recently Vahdati .S [11], J.Biazar [13], and B.Sita Ram Babu [1] successfully applied this technique to solve different types of non-linear problems. Inspired from that we propose the HAM method to investigate the series solutions of the three species ecological model with prey, predator and a competitor to both the prey and predator .The HAM itself provides us with a convenient way to control adjust the convergence region and rate of approximation series.

A. Basic equations:

The governing equations of the system are as follows

$$\begin{aligned}
\frac{dN_1}{dt} &= a_1N_1 - \alpha_{11}N_1{}^2 - \alpha_{12}N_1N_2 - \alpha_{13}N_1N_3 \\
\frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2{}^2 + \alpha_{21}N_1N_2 - \alpha_{23}N_2N_3 \qquad (1.1) \\
\frac{dN_3}{dt} &= a_3N_3 - \alpha_{33}N_3^2 - \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3
\end{aligned}$$

Where N_i , N_2 and N_3 are the populations of the prey and predator and a competitor to the predator with the natural growth rates a_1 , a_2 and a_3 respectively,

α_{11} is rate of decrease of the prey due to insufficient food and inter species competition,

α_{12} is rate of decrease of the prey due to inhibition by the predator,

α_{21} is rate of increase of the predator due to successful attacks on the prey,

α_{22} is rate of decrease of the predator due to insufficient food other than the prey and inter species competition,

α_{23} is rate of decrease of the predator due to the competition with the third species

α_{33} is rate of decrease of the Competitor to the both prey and predator due to insufficient food and inter species competition,

α_{32} is rate of decrease of the competitor due to the competition with the predator,

α_{13} is rate of decrease of the prey due to the competition with the competitor ,

α_{31} is rate of decrease of the competitor due to the competition with the prey .

II. BASIC IDEAS OF HAM

In this paper, we apply the homotopy analysis method to the discussed problem. To show the basic idea, let us consider the following differential equation N (v (t), t) =0, (2.1)
Where N is a nonlinear operator, t denote independent variable, v (t) is an unknown function, By means of generalizing the traditional homotopy method constructs the zero-order deformation

equation $(1-q)L[\psi(t;q)-v_0(t)] = q\bar{h}H(t)N[\psi(t;q)],$ (2.2)

Where $q \in [0,1]$ is the embedding parameter , $\bar{h}$ is a nonzero auxiliary, parameter H is an auxiliary ,parameter H is an auxiliary function L is an auxiliary linear operator, $v_0(t)$ is an initial guess of v(t),ψ(t, q) is a unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously,
When q=0 and q=1, it holds $\psi(t, 0) = v_0(t)$, $\psi(t, 1) = v(t)$ (2.3)
Respectively. Thus as q increases from 0 to 1, the solution ψ (t; q) varies from the initial guesses v_0 (t) to the solution v (t). Expanding ψ (t; q) in Taylor series with respect to q, one has

$$\psi(t;q) = v_0(t) + \sum_{m=1}^{+\infty} v_m(t) q^m \qquad (2.4)$$

Where $$v_m(t) = \frac{1}{m!} \frac{\partial^m \psi(t;q)}{\partial q^m}\bigg|_{q=0} \qquad (2.5)$$

If the initial guess (2.3), the auxiliary linear parameter L_i, the non-zero auxiliary parameter h_i and the auxiliary function H_i are properly chosen that the power series (2.4) converges at p=1.
Then we have under these assumptions the solution series

$$v(t) = v_0(t) + \sum_{m=1}^{+\infty} v_m(t) \qquad (2.6)$$

Which must be one of solution s of original non linear equation, as proved by Liao As $\bar{h}$ =-1 and
H (t) =1, Eq (2.2) becomes $(1-q)L[\psi(t;q)-v_0(t)] + qN[\psi(t;q)] = 0$ (2.7)

Define the vector $\bar{v}_k = \{\wedge v_{0,} v_1, ...v_k\}$ Differentiating Eq. (2.7), m times with respect to embedding parameter p and then setting p=0 and finally dividing them by m! , we have the so-called m[th] order deformation equation

$$L[v_m(t) - \chi_m v_{m-1}(t)] = \bar{h}H(t)R_m(\vec{v}_{m-1}), \qquad (2.8)$$

Where $$R_m(\vec{v}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\psi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \ \& \ \chi_m = \begin{cases} 0, m \le 1, \\ 1, m > 1. \end{cases}$$

III. HAM SOLUTION FOR MODEL (1.1)

Consider the nonlinear differential equation (1.1) with initial conditions .we assume the solution of the system (1.1), $N_1(t)$, $N_2(t)$, $N_3(t)$ can be expressed by following set of base functions in the form

$$N_1(t) = \sum_{m=1}^{+\infty} a_m t^m,\ N_2 = \sum_{m=1}^{+\infty} b_m t^m, N_3(t) = \sum_{m=1}^{+\infty} c_m t^m \qquad (3.1)$$

Where a_m, b_m ,c_m are coefficients to be determined. This provides us the so called rule of solution expression i.e., the solution of (1.1) must be expressed in the same from as (3.1) .According to (1.1) and (3.1) we chose the linear operator to solve the system of Eqs.(1.1),Homotopy analysis method is employed. We consider the following initial approximations

$$\begin{aligned} N_1(t) &= N_1(t=0) = N_{10} \\ N_2(t) &= N_2(t=0) = N_{20} \\ N_3(t) &= N_3(t=0) = N_{30} \end{aligned} \tag{3.2}$$

The linear and non-linear terms are denoted as follows.

$$L_1[N_1(t;q)] = \frac{dN_1(t;q)}{dt}, L_2[N_2(t;q)] = \frac{dN_2(t;q)}{dt}, L_3[N_3(t;q)] = \frac{dN_3(t;q)}{dt}, \tag{3.3}$$

$$N_1(t,q) = \frac{dN_1(t;q)}{dt} - a_1 N_1 + \alpha_{11} N_1^{\ 2} + \alpha_{12} N_1 N_2 + \alpha_{13} N_1 N_3 \tag{3.4}$$

$$N_2(t,q) = \frac{dN_2(t;q)}{dt} - a_2 N_2 + \alpha_{22} N_2^{\ 2} - \alpha_{21} N_1 N_2 + \alpha_{23} N_2 N_3 \tag{3.5}$$

$$N_3(t,q) = \frac{dN_3(t;q)}{dt} - a_3 N_3 + \alpha_{33} N_3^2 + \alpha_{31} N_1 N_3 + \alpha_{32} N_2 N_3 \tag{3.6}$$

Using above definition the zero order deformation equation can be constructed

$$\begin{aligned} (1-q)L_1[N_1(t;q) - N_{10}(t)] &= qh_1 N_1[N_1, N_2, N_3], \\ (1-q)L_2[N_2(t;q) - N_{20}(t)] &= qh_2 N_2[N_1, N_2, N_3], \\ (1-q)L_3[N_3(t;q) - N_{30}(t)] &= qh_3 N_3[N_1, N_2, N_3]. \end{aligned} \tag{3.7}$$

When q=0 and q=1, from the zero-deformation equations one has,

$$\begin{aligned} &N_1(t;0) = N_{10}(t) \quad N_1(t;1) = N_1(t) \\ &N_2(t;0) = N_{20}(t) \quad N_2(t;1) = N_2(t) \\ &N_3(t;0) = N_{30}(t) \quad N_3(t;1) = N_3(t) \end{aligned} \tag{3.8}$$

Expand $N_1(t;q)$, $N_2(t;q)$ and $N_3(t;q)$ in Taylors series, with respect to embedding parameter q. we obtain

$$\begin{aligned} N_1(t;q) &= N_{10}(t) + \sum_{m=1}^{+\infty} N_{1m}(t) q^m \\ N_2(t;q) &= N_{20}(t) + \sum_{m=1}^{+\infty} N_{2m}(t) q^m \\ N_3(t;q) &= N_{30}(t) + \sum_{m=1}^{+\infty} N_{3m}(t) q^m \end{aligned} \tag{3.9}$$

$$\begin{aligned} N_{1m}(t) &= \frac{1}{m!} \frac{d^m N_1(t;q)}{dq^m}\bigg|_{q=0} \\ N_{2m}(t) &= \frac{1}{m!} \frac{d^m N_2(t;q)}{dq^m}\bigg|_{q=0} \\ N_{3m}(t) &= \frac{1}{m!} \frac{d^m N_3(t;q)}{dq^m}\bigg|_{q=0} \end{aligned} \tag{3.10}$$

$$q=1\begin{cases} N_1(t)=N_{10}(t)+\sum_{m=1}^{+\infty}N_{1m}(t) \\ N_2(t)=N_{20}(t)+\sum_{m=1}^{+\infty}N_{2m}(t) \\ N_3(t)=N_{30}(t)+\sum_{m=1}^{+\infty}N_{3m}(t) \end{cases} \quad (3.11)$$

Define the vector

$$N_1=[N_{10}(t),N_{11}(t),....N_{1m}(t)],N_2=[N_{20}(t),N_{21}(t),....N_{2m}(t)],N_3=[N_{30}(t),N_{31}(t),....N_{3m}(t)] \quad (3.12)$$

And apply the procedure stated before. The following m[th]-order deformation Eq will be achieved.

$$L_1[N_{1m}(t)-\chi_m N_{1,m-1}(t)]=\bar{h}_1 H_1(t)R_{1m}(N_{1,m-1},N_{2,m-1},N_{3,m-1}),$$
$$L_2[N_{2m}(t)-\chi_m N_{2,m-1}(t)]=\bar{h}_2 H_2(t)R_{2m}(N_{1,m-1},N_{2,m-1},N_{3,m-1}), \quad (3.13)$$
$$L_3[N_{3m}(t)-\chi_m N_{3,m-1}(t)]=\bar{h}_3 H_3(t)R_{3m}(N_{1,m-1},N_{2,m-1},N_{3,m-1})$$

Let us consider $H_1(t)=H_2(t)=1$ and the initial conditions are from (3.2)

$$R_{1m}(N_{1,m-1},N_{2,m-1},N_{3,m-1})=\frac{1}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}N_1[(t,q)]$$
$$=\frac{d}{dt}N_{1,m-1}(t)-a_1N_{1,m-1}+\alpha_{11}\sum_{n=1}^{m}N_{1,n}(t)N_{1,m-n-1}(t)+\alpha_{12}\sum_{n=0}^{m-1}N_{1,n}(t)N_{2,m-n-1}(t)+\alpha_{13}\sum_{n=0}^{m-1}N_{1,n}(t)N_{3,m-n-1}(t)$$
$$R_{2m}(N_{1,m-1},N_{2,m-1},N_{3,m-1})=\frac{1}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}N_2[(t,q)]$$
$$=\frac{d}{dt}N_{2,m-1}(t)-a_2N_{2,m-1}+\alpha_{22}\sum_{n=1}^{m}N_{2,n}(t)N_{2,m-n-1}(t)-\alpha_{21}\sum_{n=1}^{m}N_{2,n}(t)N_{1,m-n-1}(t)+\alpha_{23}\sum_{n=1}^{m}N_{2,n}(t)N_{3,m-n-1}(t)$$
$$R_{3m}(N_{1,m-1},N_{2,m-1},N_{3,m-1})=\frac{1}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}N_3[(t,q)]$$
$$=\frac{d}{dt}N_{3,m-1}(t)-a_3N_{3,m-1}+\alpha_{33}\sum_{n=1}^{m}N_{3,n}(t)N_{3,m-n-1}(t)+\alpha_{31}\sum_{n=0}^{m-1}N_{3,n}(t)N_{1,m-n-1}(t)+\alpha_{32}\sum_{n=0}^{m-1}N_{3,n}(t)N_{2,m-n-1}(t)$$

(3.14)

The following will be obtained successively $N_{11}=h_1k_1t, where\,k_1=\left[-a_1N_{10}+\alpha_{11}N_{10}^2+\alpha_{12}N_{10}N_{20}+\alpha_{13}N_{10}N_{30}\right]$

(3.15)

$$N_{21}=h_2k_2t, where k_2=\left[-a_2N_{20}+\alpha_{22}N_{20}^2-\alpha_{21}N_{10}N_{20}+\alpha_{23}N_{20}N_{30}\right] \quad (3.16)$$
$$N_{31}=h_3k_3t, where\,k_3=\left[-a_3N_{30}+\alpha_{33}N_{30}^2+\alpha_{31}N_{10}N_{30}+\alpha_{32}N_{20}N_{30}\right] \quad (3.17)$$

$$N_{12}=(h_1+h_1^2)k_1t+\frac{t^2}{2}l_1$$
$$where\,l_1=-a_1h_1^2k_1+2\alpha_{11}h_1^2k_1N_{10}+\alpha_{12}h_1^2k_1N_{20}+\alpha_{12}h_1h_2k_2N_{10}+\alpha_{13}h_1^2k_1N_{30}+\alpha_{13}h_1h_3k_3N_{10}$$
$$N_{22}=(h_2+h_2^2)k_2t+\frac{t^2}{2}l_2$$
$$where l_2=-a_2h_2^2k_2+2\alpha_{22}h_2^2k_2N_{20}-\alpha_{21}h_2^2k_2N_{10}-\alpha_{21}h_1h_2k_1N_{20}+\alpha_{23}h_2^2k_2N_{30}+\alpha_{23}h_3h_2k_3N_{20}$$
$$N_{32}=(h_3+h_3^2)k_3t+\frac{t^2}{2}l_3$$
$$where l_3=-a_3h_3^2k_3+2\alpha_{33}h_3^2k_3N_{30}+\alpha_{31}h_3^2k_3N_{10}+\alpha_{31}h_1h_3k_1N_{30}+\alpha_{32}h_3^2k_3N_{20}+\alpha_{32}h_2h_3k_2N_{30}$$

(3.18)

$$N_{13} = (1+h_1)(h_1+h_1^2)k_1 t + \frac{t^2}{2} m_1 + \frac{t^3}{3} p_1$$

where

$$m_1 = \begin{bmatrix} (1+h_1)l_1 - a_1 h_1 (h_1+h_1^2)k_1 + 2\alpha_{11} h_1 N_{10} (h_1+h_1^2)k_1 + \alpha_{12} h_1 N_{20} (h_1+h_1^2)k_1 + \alpha_{12} h_1 N_{10} (h_2+h_2^2)k_2 \\ +\alpha_{13} h_1 N_{30} (h_1+h_1^2)k_1 + \alpha_{13} h_1 N_{10} (h_3+h_3^2)k_3 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} -\frac{1}{2} a_1 h_1 l_1 + \frac{1}{2} h_1 \alpha_{12} l_1 N_{20} + \alpha_{11} h_1^3 k_1^2 + h_2 \alpha_{12} h_1^2 k_1 k_2 + h_1 N_{10} \alpha_{11} l_1 + \frac{1}{2} h_1 \alpha_{12} N_{10} l_2 \\ +\frac{1}{2} h_1 \alpha_{13} N_{30} l_1 + \frac{1}{2} h_1 \alpha_{13} N_{10} l_3 + h_3 \alpha_{13} h_1^2 k_1 k_3 \end{bmatrix} \quad (3.19)$$

$$N_{23} = (1+h_2)(h_2+h_2^2)k_2 t + \frac{t^2}{2} m_2 + \frac{t^3}{3} p_2$$

$$m_2 = \begin{bmatrix} (1+h_2)l_2 - a_2 h_2 (h_2+h_2^2)k_2 + 2\alpha_{22} h_2 N_{20} (h_2+h_2^2)k_2 - \alpha_{21} h_2 N_{20} (h_1+h_1^2)k_1 \\ -\alpha_{21} h_2 N_{10} (h_2+h_2^2)k_2 + \alpha_{23} h_2 N_{20} (h_3+h_3^2)k_3 + \alpha_{23} h_2 N_{30} (h_2+h_2^2)k_2 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} -\frac{1}{2} a_2 h_2 l_2 - \frac{1}{2} h_2 \alpha_{21} l_1 N_{20} + \alpha_{22} h_2^3 k_2^2 - h_1 \alpha_{21} h_2^2 k_1 k_2 + \frac{1}{2} h_2 N_{20} \alpha_{22} l_2 - \frac{1}{2} h_2 \alpha_{21} l_2 N_{10} \\ +h_3 \alpha_{23} h_2^2 k_3 k_2 + \frac{1}{2} h_2 \alpha_{23} l_2 N_{30} + \frac{1}{2} h_2 \alpha_{23} l_3 N_{20} \end{bmatrix} \quad (3.20)$$

$$N_{33} = (1+h_3)(h_3+h_3^2)k_3 t + \frac{t^2}{2} m_3 + \frac{t^3}{3} p_3$$

where

$$m_3 = \begin{bmatrix} (1+h_3)l_3 - a_3 h_3 (h_3+h_3^2)k_3 + 2\alpha_{33} h_3 N_{30} (h_3+h_3^2)k_3 + \alpha_{31} h_3 N_{30} (h_1+h_1^2)k_1 \\ +\alpha_{31} h_3 N_{10} (h_3+h_3^2)k_3 + \alpha_{32} h_3 N_{30} (h_2+h_2^2)k_2 + \alpha_{32} h_3 N_{20} (h_3+h_3^2)k_3 \end{bmatrix} \quad (3.21)$$

$$p_3 = \begin{bmatrix} -\frac{1}{2} a_3 h_3 l_3 + \alpha_{33} h_3^3 k_3^2 + h_3 N_{30} \alpha_{33} l_3 + \frac{1}{2} h_3 \alpha_{31} N_{30} l_1 + h_1 \alpha_{31} h_3^2 k_3 k_1 + \frac{1}{2} h_3 \alpha_{31} l_3 N_{30} \\ \frac{1}{2} h_3 \alpha_{32} N_{20} l_3 + h_2 \alpha_{32} h_3^2 k_3 k_2 + \frac{1}{2} h_3 \alpha_{32} l_2 N_{30} \end{bmatrix}$$

The three terms approximation to the solution will be considered as

$$\begin{aligned} N_1(t) &\approx N_{10} + N_{11} + N_{12} + N_{13} \\ N_2(t) &\approx N_{20} + N_{21} + N_{22} + N_{23} \\ N_3(t) &\approx N_{30} + N_{31} + N_{32} + N_{33} \end{aligned} \quad (3.22)$$

$$\begin{aligned} N_1(t) &\approx N_{10} + \left[3h_1 + 3h_1^2 + h_1^3\right]k_1 t + (l_1+m_1)\frac{t^2}{2} + \frac{t^3}{3} p_1 \\ N_2(t) &\approx N_{20} + \left[3h_2 + 3h_2^2 + h_2^3\right]k_2 t + (l_2+m_2)\frac{t^2}{2} + \frac{t^3}{3} p_2 \\ N_3(t) &\approx N_{30} + \left[3h_3 + 3h_3^2 + h_3^3\right]k_3 t + (l_3+m_3)\frac{t^2}{2} + \frac{t^3}{3} p_3 \end{aligned} \quad (3.23)$$

IV. **NUMERICAL SIMULATION**

Here we choose a set of parametric values for the mathematical model (1.1) and computed the approximate solutions of N_1, N_2 and N_3 by ham method up to three approximations as the number of approximation are more the results are coincide with exact solutions .we calculate exact solution by RK method and graphs are compared with Ham results .Let us assume the following values for the model (1.1)

Example 1. N_{10}=20; N_{20}=20; N_{30}=5; a_1=1; a_2=1; a_3=1.5; α_{11}=0.2; α_{12}=0.1; α_{13}=0.1; α_{22}=0.2; α_{21}=.1; α_{33}=0.2; α_{31}=.1; α_{32}=.2; α_{23}=.1 h_1=-0.025; h_2=-0.025; h_3=-0.03.

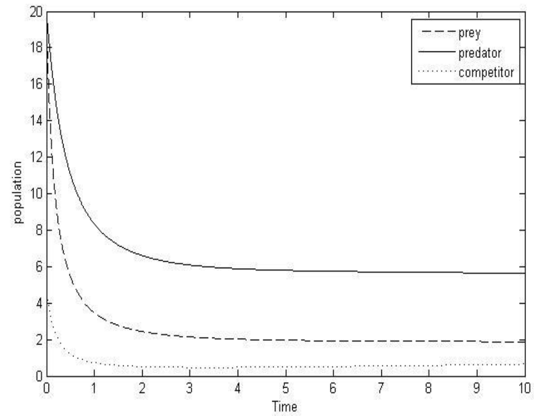


Fig 4.1: The variation of N_1, N_2 & N_3 with respective Time (t) by RK Method

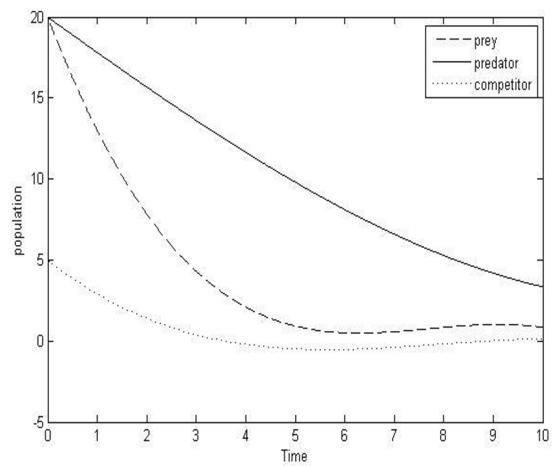
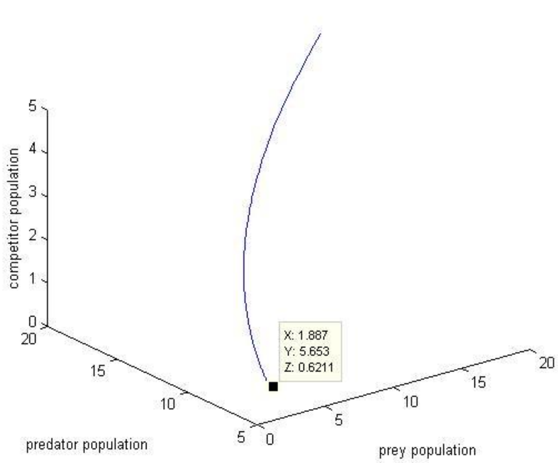


Fig 4.2: The variation of N_1, N_2 & N_3 with respective Time (t) by HAM Method



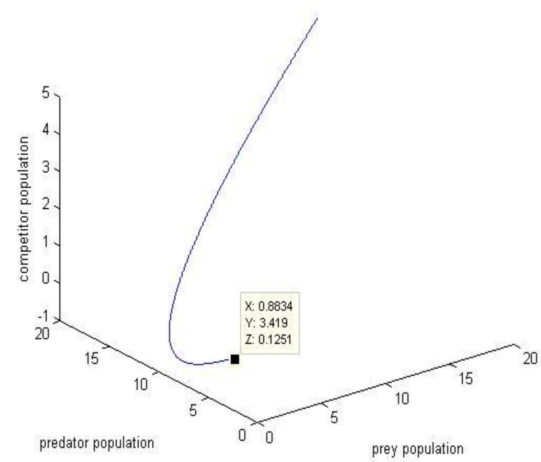


Fig 4.3 **Fig 4.4**

Fig 4.3 & 4.4: the Phage portrait of N_1, N_2, N_3

We study the variation of N_1, N_2 & N_3 with respective the time (t) are shown in fig 4.1 & 4.2

The fig 4.1 shows the graph for system of equations (1.1) solved by numerical method (RK Method) which gives exact solutions, where as we find the series approximation for the system of equations (1.1) by HAM method up to third order approximation. Similarly fig 4.3 & 4.4 shows the phase portrait N_1, N_2 & N_3 by RK method and HAM method respectively. Fig 4.5 shows the phase portrait of HAM method and numerical solution method.

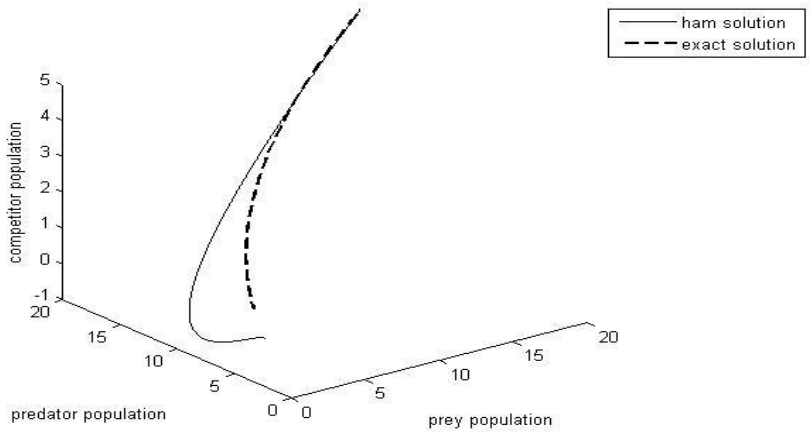


Fig 4.5

The comparative study shows that series approximations of equations (1.1) by HAM method are almost agree with the exact solutions. Computation of higher order of approximation, the solutions obtained by HAM method converge towards an exact solution. The third order approximations by ham almost agree with exact solutions.

V. CONCLUSION

We proposed the HAM technique for the three species ecological model with a prey, predator and a competitor to both the prey and predator we approximate the series solution upto third approximation. The third order approximations by ham almost agree with exact solutions. The higher order of approximation of series solutions of HAM coincide with the exact solutions.

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