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Proof of Fifth Standard Form of Nonelementary Functions

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Abstract*—*In this paper we have proved the fifth standard form of nonelementary functions introduced by Yadav & Sen by the help of strong Liouville's theorem and its corollaries. Every nonelementary functions discussed in the paper is of trigonometric or hyperbolic type functions and the Frensel integrals are their particular case. AMS Subject Classification 2010: 97I50, 26A09

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I. Introduction

The first example which leads us beyond the region of elementary functions is the *elliptic integrals* due to *John Wallis* (1655). Such integrals cannot be evaluated in terms of the elementary functions was proved by *Joseph Liouville* in 1833. The main results on functions with nonelementary integrals began with *Liouville's* results. *Marchisotto & Zakeri* [2] mentioned two examples 4 and 5 [2] obtained from the special case of strong Liouville theorem, which play a major role in proving different functions as nonelementary.

II. Fifth Standard Form Of Nonelementary Functions

Yadav & Sen [4] have introduced six standard forms of nonelementary functions as indefinite nonintegrable functions and the fifth form is as follows:

"An indefinite integral of the form $\int g[f(x)]dx$, where f(x) is a polynomial of degree greater than or equal to 2 and g(x) is a trigonometric (not inverse trigonometric) or a hyperbolic (not inverse hyperbolic) function is always nonelementary."

Proof: We will prove it in three different cases:
Case I: **When g(x) is a trigonometric (not inverse trigonometric) function**. Then

2.**1. For g(x)=sinx,** we have

$$\int \sin f(x)dx = \int \frac{e^{if(x)} - e^{-if(x)}}{2i}dx = \frac{1}{2i}\left[\int e^{if(x)}dx - \int e^{-if(x)}dx\right]$$

By strong Liouville theorem (special case), the first integral is elementary if and only if there exists a rational function R(x) which satisfies an identity of the form

$$1 = R'(x) + iR(x)f'(x) \Rightarrow R'(x) = 1 \text{ and } R(x)f'(x) = 0$$
$$\Rightarrow R(x) = 0 \text{ or } f'(x) = 0 \text{ and } R'(x) = 1$$

But $f'(x) \neq 0$, which implies that $R(x) = 0$ and $R'(x) = 1$, which cannot be true i. e. such R(x) does not exist. So this integral is nonelementary. Similarly we can prove that the second integral is nonelementary. Therefore the given indefinite integral is nonelementary.

2.2. For g(x)=cosx, we have

$$\int \cos f(x)dx = \int \frac{e^{if(x)} + e^{-if(x)}}{2}dx = \frac{1}{2}\left[\int e^{if(x)}dx + \int e^{-if(x)}dx\right]$$

Both indefinite integrals are nonelementary functions proved in section 2.1.
2.3. For g(x)=tanx, we have on putting secf(x)=z

$$\int \tan f(x)dx = \int \frac{\sec f(x).\tan f(x).f'(x)}{\sec f(x).f'(x)}dx = \int \frac{dz}{zf'(x)} \qquad (2.3.1)$$

For $f(x)=x^2+bx+c$, we have from (2.3.1)

$$\int\frac{dz}{zf'(x)}=\int\frac{dz}{2z\sqrt{\sec^{-1}z+k}},\ \left(k=\frac{b^2-4c}{4}\right)=\frac{1}{2}\int\frac{\sqrt{z^2-1}dz}{z\sqrt{z^2-1}\sqrt{\sec^{-1}z+k}}$$

From strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{1}{2z\sqrt{\sec^{-1}z+k}}=U_0^{'}+\sum_{i=1}^{n}C_i\frac{U_i^{'}}{U_i}$$

Considering different possible forms of U_j, like $\sqrt{\sec^{-1}z+k}$, we find that no such U_j exist. Hence the given indefinite integral is nonelementary.

2.4. For g(x)=cotx, we have on putting cosecf(x)=z

$$\int\cot f(x)dx=\int\frac{\operatorname{cosec}f(x).\cot f(x).f'(x)}{\operatorname{cosec}f(x).f'(x)}dx=-\int\frac{dz}{zf'(x)} \quad (2.4.1)$$

For $f(x)=x^2+bx+c$, we have from (2.4.1)

$$\int\frac{-dz}{zf'(x)}=\int\frac{-dz}{z\sqrt{\operatorname{cosec}^{-1}z+k}}=\int\frac{-\sqrt{z^2-1}dz}{z\sqrt{z^2-1}\sqrt{\operatorname{cosec}^{-1}z+k}}$$

which can now be proved nonelementary as has been done in section 2.3.

2.5. For g(x)=cosecx, we have on putting $e^{if(x)}=z$

$$\int\operatorname{cosec}f(x)dx=\int\frac{dx}{\sin f(x)}=\int\frac{2idx}{e^{if(x)}-e^{-if(x)}}=2\int\frac{dz}{f'(x)(z^2-1)} \quad (2.5.1)$$

For $f(x)=x^2+bx+c$, we have from (2.5.1)

$$\int\frac{2dz}{f'(x)(z^2-1)}=\frac{1}{2\sqrt{-i}}\left[\int\frac{dz}{\sqrt{\log z+k}(z-1)}-\int\frac{dz}{\sqrt{\log z+k}(z+1)}\right] \quad (2.5.2)$$

Now the first integral is elementary if and only if there exists an identity of the form

$$\frac{1}{(z-1)\sqrt{\log z+k}}=U_0^{'}+\sum_{i=1}^{n}C_i\frac{U_i^{'}}{U_i}$$

But no such U_j exist. Hence it is nonelementary. Similarly the second integral can be proved nonelementary. Therefore from (2.5.2), the given indefinite integral is nonelementary.

2.6. For g(x)=secx, we have on putting $e^{if(x)}=z$

$$\int\sec f(x)dx=\int\frac{dx}{\cos f(x)}=\int\frac{2dx}{e^{if(x)}+e^{-if(x)}}=\frac{2}{i}\int\frac{dz}{f'(x)(z^2+1)} \quad (2.6.1)$$

For $f(x)=x^2+bx+c$, it becomes

$$=\frac{1}{i\sqrt{-i}}\left[\int\frac{dz}{\sqrt{\log z+k}(z^2+1)}\right]=\frac{1}{2i\sqrt{-i}}\left[\int\frac{dz}{(1-iz)\sqrt{\log z+k}}+\int\frac{dz}{(1+iz)\sqrt{\log z+k}}\right]$$

Both can be proved nonelementary by the similar procedures as has been applied in section 2.5. Hence the given function is nonelementary.

Case II: When g(x) is a hyperbolic (not inverse hyperbolic) function. Then

2.7. For g(x)=sinhx, we have

$$\int\sinh f(x)dx=\int\frac{e^{f(x)}-e^{-f(x)}}{2}dx=\frac{1}{2}\left[\int e^{f(x)}dx-\int e^{-f(x)}dx\right] \quad (2.7.1)$$

Now by strong Liouville theorem (special case) the first integral is elementary if and only if there exists a rational function R(x) such that

$$1=R'(x)+R(x)f'(x) \quad (2.7.2)$$

Let $R(x)=\frac{p(x)}{q(x)}$, where g.c.d.{p(x), q(x)}=1. Then from (2.7.2) we get

$$1=\frac{q(x)p'(x)-p(x)q'(x)}{[q(x)]^2}+\frac{p(x)}{q(x)}f'(x)\Rightarrow\{p'(x)-q(x)+p(x)f'(x)\}=\frac{p(x)q'(x)}{q(x)}$$

$$\Rightarrow q(x)\big|p(x) \text{ or } q(x)\big|q'(x)$$

But q(x) cannot divide p(x), which implies that $q(x)\big|q'(x)$ and so q(x) is a constant. Without loss of generality we can take R(x)=p(x), a polynomial. Therefore from (2.7.2), we get

$$1=p'(x)+p(x)f'(x)$$

Comparing the degrees of x on both sides now results out in a contradiction i. e., such R(x) does not exist. Therefore this indefinite integral is nonelementary. Similarly we can prove that the second integral in (2.7.1) is nonelementary. Hence the given function is nonelementary.

2.8. For g(x)=coshx, we have

$$\int\cosh f(x)dx=\int\frac{e^{f(x)}+e^{-f(x)}}{2}dx=\frac{1}{2}\left[\int e^{f(x)}dx+\int e^{-f(x)}dx\right]$$

Both are nonelementary functions proved in section 2.7.

2.9. For g(x)=tanhx, we have on putting sechf(x)=z

$$\int\tanh f(x)dx=\int\frac{\operatorname{sech}f(x)\tanh f(x)f'(x)}{f'(x)\operatorname{sech}f(x)}dx=-\int\frac{dz}{zf'(x)} \tag{2.9.1}$$

For $f(x)=x^2+bx+c$, we have from (2.9.1)

$$\int\frac{-dz}{zf'(x)}=\int\frac{-dz}{2z\sqrt{\operatorname{sech}^{-1}z+k}}=\int\frac{-\sqrt{1-z^2}dz}{2z\sqrt{1-z^2}\sqrt{\operatorname{sech}^{-1}z+k}}$$

which can now be proved nonelementary as has been done in section 2.3.

2.10. For g(x)=cothx, we have on putting cosechf(x)=z

$$\int\coth f(x)dx=\int\frac{\operatorname{cosech}f(x)\coth f(x)f'(x)}{f'(x)\operatorname{cosech}f(x)}dx=\int\frac{-dz}{zf'(x)} \tag{2.10.1}$$

For $f(x)=x^2+bx+c$, we have from (2.10.1)

$$\int\frac{-dz}{zf'(x)}=\int\frac{-dz}{2z\sqrt{\operatorname{cosech}^{-1}z+k}}=\int\frac{-\sqrt{z^2+1}dz}{2z\sqrt{z^2+1}\sqrt{\operatorname{cosech}^{-1}z+k}}$$

which can now be proved nonelementary as has been done in section 2.3.

2.11. For g(x)=cosechx, we have on putting $e^{f(x)}=z$

$$\int\operatorname{cosech}f(x)dx=\int\frac{dx}{\sinh f(x)}=\int\frac{2dx}{e^{f(x)}-e^{-f(x)}}=2\int\frac{dz}{f'(x)(z^2-1)} \tag{2.11.1}$$

For $f(x)=x^2+bx+c$, we have from (2.11.1)

$$\int\frac{2dz}{f'(x)(z^2-1)}=\int\frac{dz}{\sqrt{\log z+k}(z^2-1)}=\frac{1}{2}\left[\int\frac{dz}{(z-1)\sqrt{\log z+k}}-\int\frac{dz}{(z+1)\sqrt{\log z+k}}\right]$$

Both integrals are nonelementary proved in section 2.5. Therefore the given indefinite integral is nonelementary.

2.12. For g(x)=sechx, we have on putting $e^{f(x)}=z$

$$\int\operatorname{sech}f(x)dx=\int\frac{dx}{\cosh f(x)}=\int\frac{2dx}{e^{f(x)}+e^{-f(x)}}=2\int\frac{dz}{f'(x)(z^2+1)} \tag{2.12.1}$$

For $f(x)=x^2+bx+c$, we have from (2.12.1)

$$\int\frac{2dz}{f'(x)(z^2+1)}=\frac{1}{2}\left[\int\frac{dz}{(1-iz)\sqrt{\log z+k}}+\int\frac{dz}{(1+iz)\sqrt{\log z+k}}\right]$$

Both indefinite integrals are nonelementary proved in section 2.5. Therefore the given indefinite integral is nonelementary.

Similarly we can prove the above integrals nonelementary for higher degree polynomial f(x).
Let us consider some examples based on this form:

Example 1: Show that the integral $\int \sin(x^2+3)dx$ is nonelementary.

Proof: We have

$$\int \sin(x^2+3)dx = \frac{1}{2i}\left[\int e^{i(x^2+3)}dx - \int e^{-i(x^2+3)}dx\right]$$

From strong Liouville theorem (special case), the first integral is elementary if and only if there exists a rational function R(x) which satisfies an identity of the form

$$1 = R'(x) + i2xR(x) \Rightarrow 1 = R'(x) \text{ and } 2xR(x) = 0$$

But x and R(x) cannot be zero, so such R(x) does not exist. Hence the first integral is nonelementary. Similarly we can prove that the second integral is nonelementary. Therefore the given integral is nonelementary.

Example 2: Show that the integral $\int \cosh(6x^2+b)dx$ is nonelementary.

Proof: We have

$$\int \cosh(6x^2+b)dx = \frac{1}{2}\left[\int e^{(6x^2+b)}dx + \int e^{-(6x^2+b)}dx\right]$$

Both can be proved nonelementary by the similar procedures as has been applied in example 1.

Example 3: Show that the integral $\int \tan(x^2+bx+c)dx$ is nonelementary.

We have

$$\int \tan(x^2+bx+c)dx = \int \frac{(2x+b)\sec(x^2+bx+c)\tan(x^2+bx+c)dx}{(2x+b)\sec(x^2+bx+c)}$$

On putting $\sec(x^2+bx+c)=z$, it becomes

$$= \int \frac{dz}{(2x+b)z} = \frac{1}{2}\int \frac{\sqrt{z^2-1}dz}{z\sqrt{z^2-1}\sqrt{\sec^{-1}z+k}}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{1}{z\sqrt{\sec^{-1}z+k}} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given integral is nonelementary.

Example 4: Show that the integral $\int \tan(3x^3+bx^2+cx+d)dx$ is nonelementary.

Proof: We have on putting sec(3x3+bx2+cx+d)=z

$$\int \tan(3x^3+bx^2+cx+d)dx = \int \frac{(9x^2+2bx+c)\sec(3x^3+bx^2+cx+d)\tan(3x^3+bx^2+cx+d)}{(9x^2+2bx+c)\sec(3x^3+bx^2+cx+d)}dx$$

$$= \int \frac{dz}{(9x^2+2bx+c)z} = \int \frac{dz}{\left[\left(3x+\frac{b}{3}\right)^2 - \left\{\left(\frac{b}{3}\right)^2 - c\right\}\right]z}$$

$$= \frac{1}{2\beta}\int \frac{dz}{(3x+\alpha-\beta)z} - \frac{1}{2\beta}\int \frac{dz}{(3x+\alpha+\beta)z}, \text{ where } \frac{b}{3}=\alpha \text{ and } \left(\frac{b}{3}\right)^2 - c = \beta^2$$

$$= \frac{1}{6\beta}\int \frac{dz}{(x+A)z} - \frac{1}{6\beta}\int \frac{dz}{(x+B)z}, \text{ where } A = \frac{\alpha-\beta}{3}, B = \frac{\alpha+\beta}{3}$$

$$= \frac{1}{6\beta}\int \frac{\sqrt{z^2-1}dz}{(x+A)z\sqrt{z^2-1}} - \frac{1}{6\beta}\int \frac{\sqrt{z^2-1}dz}{(x+B)z\sqrt{z^2-1}}$$

Since $z=\sec(3x^3+bx^2+cx+d)$, x is a function of $\sec^{-1}z$. Then both (x+A) and (x+B) are functions of $\sec^{-1}z$. Therefore both will be of the form

$$= \int F[z, \sqrt{z^2-1}, \sec^{-1}z]dz = \int F[z, y_1, y_2]dz$$

Giving the exact values of b, c, d, we can apply the same procedure as has been applied in example 3 to prove it nonelementary.

III. Conclusion

For $f(x)=x^2$, we get the Frensel integrals from sections 2.1 and 2.2 respectively

$$\int \sin(x^2)dx \text{ and } \int \cos(x^2)dx$$

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