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Proof of Fourth Standard Form of Nonelementary Functions

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***Abstract*— This paper contains the proof of the fourth standard form of nonelementary functions introduced by Yadav & Sen by applying strong Liouville's theorem, its corollary and some properties due to Marchisotto and Zakeri. Every nonelementary functions discussed in the paper is of exponential type functions and the Gaussian integral is its particular case. AMS Subject Classification 2010: 97I50, 26A09**

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I. Introduction

A natural query in elementary calculus "*what type of functions cannot be integrated?*" was well explained by *Marchisotto and Zakeri* [2] by exposing different theorems of the pioneers of the subject like *Newton*, *Leibnitz*, *Laplace*, *Abel*, *Liouville*, *Risch*, etc. However the main results on functions with nonelementary integrals began with the *Liouville's* results like *Liouville's theorem* (1833), *Strong Liouville theorem* (1835), *Strong Liouville theorem (special case, 1835)* and its corollaries. By 1841, he had developed a theory of integration that settled the question of integration in finite terms for many important cases. Recently *Yadav & Sen* [6] have propounded such six types of functions in 2007.

II. Fourth Standard Form Of Nonelementary Functions

Yadav & Sen [6] have studied a set of functions and answered the natural query mentioned in the introduction part that we can form some standard forms of functions, which cannot be integrated in in case of indefinite integrals i. e. integrals of such type of functions cannot be elementary. They introduced six such forms out of which the fourth one is as follows:

"An indefinite integral of the form $\int e^{f(x)}dx$, where f(x) is a trigonometric (not inverse trigonometric) function, or a hyperbolic (not inverse hyperbolic) function, or a polynomial of degree greater than or equal to 2, is always nonelementary."

Proof: We shall prove it in three different cases:

Case I: **When f(x) be a polynomial of degree ≥2**.

We have by strong Liouville theorem, $\int e^{f(x)}dx$ is elementary if and only if there exists a rational function R(x) which satisfies an identity

$$R'(x)+f'(x)R(x)=1 \tag{2.1}$$

Let $R(x)=\frac{p(x)}{q(x)}$; where $\gcd(p(x),q(x))=1$. Then from (2.1), we have

$$q(x)p'(x)-p(x)q'(x)+f'(x)p(x)q(x)=[q(x)]^2 \Rightarrow p'(x)+f'(x)p(x)-q(x)=\frac{p(x)q'(x)}{q(x)}$$

But q(x) cannot divide p(x) as g.c.d.(p(x), q(x))=1. Therefore q(x) divides q'(x), which means that q(x) is a constant. Without loss of generality, we can take R(x)=p(x). Then R'(x)=p'(x) and from (2.1), we have

$$p'(x)+f'(x)p(x)=1 \tag{2.2}$$

where degree of p(x)≥1 as p(x) cannot be constant. Comparing the degrees of x on both sides of (2.2) now results in a contradiction. Hence no such R(x) exist, so the given function is nonelementary.

Case II: **When f(x) be a trigonometric (not inverse trigonometric) function and φ(x) be a polynomial of degree ≥1.** Then

2.1. For **f(x)=sinφ(x)**, we have on putting sinφ(x)=z

$$\int e^{f(x)}dx=\int e^{\sin\varphi(x)}dx=\int\frac{e^z dz}{\varphi'(x)\sqrt{1-z^2}} \tag{2.1.1}$$

Sub-case-I: For φ(x)=x+b, we have from (2.1.1)

$$\int \frac{e^z dz}{\phi'(x)\sqrt{1-z^2}} = \int \frac{e^z dz}{\sqrt{1-z^2}}$$

From strong Liouville theorem (special case), it is elementary if and only if there exists an identity

$$\frac{e^z}{\sqrt{1-z^2}} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given function is nonelementary.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (2.1.1)

$$\int \frac{e^z dz}{\phi'(x)\sqrt{1-z^2}} = \int \frac{e^z dz}{(2x+b)\sqrt{1-z^2}} = \frac{1}{2}\int \frac{e^z dz}{\sqrt{\sin^{-1} z + k}\sqrt{1-z^2}}; k = \frac{b^2-4c}{4}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{e^z}{2\sqrt{\sin^{-1} z + k}\sqrt{1-z^2}} = \left[\frac{dU_0}{dz} + \sum_{i=1}^{n} c_i \frac{U'_i}{U_i}\right]$$

But no such U_j exist. Therefore the given function is nonelementary.

Alternate Proof: Let us take the logarithmic representation of $\sin^{-1}z$ given by

$$\sin^{-1} z = -i \log[iz + \sqrt{1-z^2}]$$

For φ(x)=x+b, we get the same function and result as in sub-case-I.

For φ(x)=x^2+bx+c, we have from (2.1.1)

$$\int \frac{e^z dz}{\phi'(x)\sqrt{1-z^2}} = \int \frac{e^z dz}{(2x+b)\sqrt{1-z^2}} = \frac{1}{2i\sqrt{i}}\int \frac{e^z dz}{\sqrt{[\log\{iz+\sqrt{1-z^2}\}+ik]}\sqrt{(1-z^2)}}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{e^z}{\sqrt{\log\{iz+\sqrt{1-z^2}\}+ik}\sqrt{1-z^2}} = \frac{d}{dz}\left[U_0 + \sum_{i=1}^{n} C_i \log U_i\right]$$

But no such U_j exist. Therefore the given function is nonelementary.

2.2. For **f(x)=cosφ(x),** we have on putting cosφ(x)=z

$$\int e^{f(x)} dx = \int e^{\cos\varphi(x)} dx = \int \frac{-e^z dz}{\varphi'(x)\sqrt{1-z^2}} \qquad (2.2.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.2.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x)\sqrt{1-z^2}} = \int \frac{-e^z dz}{\sqrt{1-z^2}} \qquad (2.2.2)$$

Which is nonelementary proved in section 2.1 sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, the integral (2.2.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x)\sqrt{1-z^2}} = \int \frac{-e^z dz}{2\sqrt{\cos^{-1} z + k}\sqrt{1-z^2}} \qquad (2.2.3)$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{-e^z}{2\sqrt{\cos^{-1} z + k}\sqrt{1-z^2}} = \frac{d}{dz}\left[U_0 + \sum_{i=1}^{n} C_i \log U_i\right]$$

But such U_j does not exist. Therefore the given function is nonelementary.

2.3. For **f(x)=tanφ(x)**, we have on putting tanφ(x)=z

$$\int e^{f(x)} dx = \int e^{\tan\varphi(x)} dx = \int \frac{e^z dz}{\varphi'(x)(1+z^2)} \qquad (2.3.1)$$

Sub-case-I: For φ(x)=x+b, we have from (2.3.1)

$$\int \frac{e^z dz}{\varphi'(x)(1+z^2)} = \int \frac{e^z dz}{(1+z^2)} = \frac{1}{2}\left[\int \frac{e^z dz}{(1+iz)} + \int \frac{e^z dz}{(1-iz)}\right]$$

Now

$$\int \frac{e^z dz}{(1+iz)} = \frac{e^i}{i}\int \frac{e^{-ip}}{p} dp, \text{putting } (1+iz) = p$$

By strong Liouville theorem (special case), it is elementary if and only if there exists a rational function R(x) which satisfies the identity

$$\frac{1}{p} = R'(p) - iR(p) \Rightarrow R(p) = 0 \text{ and } R'(p) = \frac{1}{p}$$

But R(p) cannot be zero, so such R(p) does not exist. Hence it is nonelementary.

Also

$$\int \frac{e^z dz}{(1-iz)} = ie^{-i}\int \frac{e^{ip}}{p} dp, \text{putting } (1-iz) = p$$

Again by strong Liouville theorem (special case), it is elementary if and only if there exists a rational function R(x) which satisfies the identity

$$\frac{1}{p} = R'(p) + iR(p) \Rightarrow R(p) = 0 \text{ and } R'(p) = \frac{1}{p}$$

But R(p) cannot be zero, so such R(p) does not exist. Hence it is nonelementary. Therefore the given function is nonelementary.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (2.3.1)

$$\int \frac{e^z dz}{\varphi'(x)(1+z^2)} = \int \frac{e^z dz}{2\sqrt{\tan^{-1} z + k}\,(1+z^2)}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity containing U_j , a function of z, y_1, y_2, and y_3 of the form

$$\frac{e^z}{2\sqrt{\tan^{-1} z + k}\,(1+z^2)} = \frac{dU_0}{dz} + \sum_{i=1}^{n} c_i \frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given function is nonelementary.

2.4. For **f(x)=cotφ(x)**, we have on putting cotφ(x)=z

$$\int e^{f(x)} dx = \int e^{\cot\varphi(x)} dx = \int \frac{-e^z dz}{\varphi'(x)(1+z^2)} \qquad (2.4.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.4.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x)(1+z^2)} = \int \frac{-e^z dz}{(1+z^2)}$$

Which is nonelementary, proved in section 2.3, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, the integral (2.4.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x)(1+z^2)} = \int \frac{-e^z dz}{2(1+z^2)\sqrt{\cot^{-1} z + k}}$$

A similar argument will hold as in section 2.3, sub-case-II to prove it nonelementary.

2.5. For **f(x)=cosecφ(x)**, we have on putting cosecφ(x)=z

$$\int e^{f(x)} dx = \int e^{\operatorname{cosec}\varphi(x)} dx = \int \frac{-e^z dz}{\varphi'(x) z\sqrt{z^2-1}} \qquad (2.5.1)$$

Sub-case-I: For φ(x)=x+b, we have (2.5.1)

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{z^2-1}} = \int \frac{-e^z dz}{z\sqrt{z^2-1}}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{e^z}{z\sqrt{z^2-1}} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given function is nonelementary.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (2.5.1)

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{z^2-1}} = -\int \frac{e^z dz}{(2x+b) z\sqrt{z^2-1}} = -\frac{1}{2}\int \frac{e^z dz}{z\sqrt{\operatorname{cosec}^{-1} z + k}\,\sqrt{z^2-1}}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity containg U_j, a function of z, y_1, y_2, and y_3 of the form

$$\frac{e^z}{2z\sqrt{\operatorname{cosec}^{-1} z + k}\,\sqrt{z^2-1}} = \frac{dU_0}{dz} + \sum_{i=1}^{n} c_i \frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given function is nonelementary.

2.6. For **f(x)=secφ(x)**, we have on putting secφ(x)=z

$$\int e^{f(x)} dx = \int e^{\sec\varphi(x)} dx = \int \frac{e^z dz}{\varphi'(x) z\sqrt{z^2-1}} \qquad (2.6.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.6.1) becomes

$$\int \frac{e^z dz}{\varphi'(x) z\sqrt{z^2-1}} = \int \frac{e^z dz}{z\sqrt{z^2-1}}$$

Which is nonelementary, proved in section 2.5, sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, the integral (2.6.1) becomes

$$\int \frac{e^z dz}{\varphi'(x) z\sqrt{z^2-1}} = \int \frac{e^z dz}{2z\sqrt{\sec^{-1} z + k}\,\sqrt{z^2-1}}$$

Which can now be proved nonelementary by strong Liouville theorem by the similar procedures as has been applied in section 2.5, sub-case-II.

Case III: **When f(x) be a hyperbolic (not inverse hyperbolic) function and φ(x) is a polynomial of degree ≥1**. Then

2.7. For **f(x)=sinhφ(x)**, we have on putting sinhφ(x)=z

$$\int e^{f(x)} dx = \int e^{\sinh\varphi(x)} dx = \int \frac{e^z dz}{\varphi'(x)\sqrt{1+z^2}} \qquad (2.7.1)$$

Sub-case-I: For φ(x)=x+b, we have from (2.7.1)

$$\int \frac{e^z dz}{\varphi'(x)\sqrt{1+z^2}} = \int \frac{e^z dz}{\sqrt{1+z^2}}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{e^z}{\sqrt{1+z^2}} = \frac{d}{dz}\left[U_0 + \sum_{i=1}^{n} C_i \log U_i\right]$$

But no such U_j exist i.e., the given function is nonelementary.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (2.7.1)

$$\int \frac{e^z dz}{\varphi'(x)\sqrt{1+z^2}} = \int \frac{e^z dz}{2\sqrt{\sinh^{-1} z + k}\,\sqrt{1+z^2}}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{e^z}{\sqrt{\sinh^{-1} z + k}\sqrt{1+z^2}} = \frac{d}{dz}\left[U_0 + \sum_{i=1}^{n} C_i \log U_i\right]$$

But no such U_j exist. Hence the given function is nonelementary.

2.8. For **f(x)=coshφ(x)**, we have on putting coshφ(x)=z

$$\int e^{f(x)}dx = \int e^{\cosh\varphi(x)}dx = \int \frac{e^z dz}{\varphi'(x)\sqrt{z^2-1}} \quad (2.8.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.8.1) becomes

$$\int \frac{e^z dz}{\varphi'(x)\sqrt{z^2-1}} = \int \frac{e^z dz}{\sqrt{z^2-1}}$$

which can be proved nonelementary by strong Liouville theorem by the similar procedure as has been applied in section 2.2, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, the integral (2.8.1) becomes

$$\int \frac{e^z dz}{\varphi'(x)\sqrt{z^2-1}} = \int \frac{e^z dz}{2\sqrt{\cosh^{-1} z + k}\sqrt{z^2-1}}$$

Which can now be proved nonelementary by strong Liouville theorem by the similar procedures as has been applied in sections 2.2 and 2.7, sub-case-II.

2.9. For **f(x)=tanhφ(x)**, we have on putting tanhφ(x)=z

$$\int e^{f(x)}dx = \int e^{\tanh\varphi(x)}dx = \int \frac{e^z dz}{\varphi'(x)(1-z^2)} \quad (2.9.1)$$

Sub-case-I: For φ(x)=x+b, we have from (2.9.1)

$$\int \frac{e^z dz}{\varphi'(x)(1-z^2)} = \int \frac{e^z dz}{(1-z^2)} = \frac{1}{2}\left[\int \frac{e^z dz}{(1-z)} + \int \frac{e^z dz}{(1+z)}\right]$$

Where,

$$\int \frac{e^z dz}{(1-z)} = -e\int \frac{e^{-p}}{p}dp, [\text{Putting } 1-z=p] \quad \text{and} \quad \int \frac{e^z dz}{(1+z)} = \frac{1}{e}\int \frac{e^p}{p}dp, [\text{Putting } 1+z=p]$$

Both are nonelementary from example 4 [2, p.300]. Therefore the given function is also nonelementary.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, we have from (2.9.1)

$$\int \frac{e^z dz}{\varphi'(x)(1-z^2)} = \int \frac{e^z dz}{2\sqrt{\tanh^{-1} z + k}(1-z^2)}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{e^z}{\sqrt{\tanh^{-1} z + k}(1-z^2)} = \frac{d}{dz}\left[U_0 + \sum_{i=1}^{n} C_i \log U_i\right]$$

But no such U_j exist. Hence the given function is nonelementary.

2.10. For **f(x)=cothφ(x)**, we have on putting cothφ(x)=z

$$\int e^{f(x)}dx = \int e^{\coth\varphi(x)}dx = \int \frac{e^z dz}{\varphi'(x)(1-z^2)} \quad (2.10.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.10.1) becomes

$$\int \frac{e^z dz}{\varphi'(x)(1-z^2)} = \int \frac{e^z dz}{(1-z^2)}$$

which is nonelementary from example 4 [2, p.300] as discussed in section 2.9, sub-case-I.

Sub-case-II: For $\varphi(x)=x^2+bx+c$, the integral (2.10.1) becomes

$$\int \frac{e^z dz}{\varphi'(x)(1-z^2)} = \int \frac{e^z dz}{2\sqrt{\coth^{-1} z + k}(1-z^2)}$$

Which can now be proved nonelementary by strong Liouville theorem by the similar procedures as has been applied in section 2.9, sub-case-II.

2.11. **For f(x)=cosechφ(x)**, we have on putting cosechφ(x)=z

$$\int e^{f(x)}dx = \int e^{\operatorname{cosech}\varphi(x)}dx = \int \frac{-e^z dz}{\varphi'(x) z\sqrt{1+z^2}} \qquad (2.11.1)$$

Sub-case-I: For φ(x)=x+b, we have from (2.11.1)

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{1+z^2}} = \int \frac{-e^z dz}{z\sqrt{1+z^2}}$$

It can now be proved nonelementary by the similar procedures as has been applied in section 2.5, sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, we have from (2.11.1)

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{1+z^2}} = \int \frac{-e^z dz}{2z\sqrt{\operatorname{cosech}^{-1} z + k}\sqrt{1+z^2}}$$

It can now be proved nonelementary by the similar procedures as has been applied in section 2.5, sub-case-II.

2.12. For **f(x)=sechφ(x)**, we have on putting sechφ(x)=z

$$\int e^{f(x)}dx = \int e^{\operatorname{sech}\varphi(x)}dx = \int \frac{-e^z dz}{\varphi'(x) z\sqrt{1-z^2}} \qquad (2.12.1)$$

Sub-case-I: For φ(x)=x+b, the integral (2.12.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{1-z^2}} = \int \frac{-e^z dz}{z\sqrt{1-z^2}}$$

which can now be proved nonelementary by the similar procedures as has been given in section 2.5, sub-case-I.

Sub-case-II: For φ(x)=x^2+bx+c, the integral (2.12.1) becomes

$$\int \frac{-e^z dz}{\varphi'(x) z\sqrt{1-z^2}} = \int \frac{-e^z dz}{2z\sqrt{\operatorname{sech}^{-1} z + k}\sqrt{1-z^2}}$$

Which can now be proved nonelementary by strong Liouville theorem by the similar procedures as has been applied in section 2.5, sub-case-II.

Similarly we can extend the proof for higher degree polynomials φ(x). In the above proofs we can take the logarithmic representations of inverse hyperbolic functions to prove them nonelementary as has been done in section 2.1.

Let us consider some nonelementary functions based on the fourth standard type:

Example 1: Show that the integral $\int e^{x^2} dx$ is nonelementary.

Proof: It is a well defined nonelementary function from example 7 [2, p.302].

Example 2: Show that the integral $\int e^{-x^2} dx$ is nonelementary.

Proof: It is a well defined nonelementary function from example 7 [2, p.302].

Example 3: Show that the integral $\int e^{\sin x} dx$ is nonelementary.

Proof: On putting sinx=z, we have

$$\int e^{\sin x}dx = \int \frac{e^z dz}{\sqrt{1-z^2}}$$

Which is nonelementary, proved in section 2.1 for φ(x)=x+b in sub-case-I.

Example 4: Show that the integral $\int e^{\tan x} dx$ is nonelementary.

Proof: On putting tanx=z, we get

$$\int e^{\tan x} dx = \int \frac{e^z dz}{(1+z^2)}$$

Which is nonelementary, proved in section 2.3, sub-case-I.

Example 5: Show that the integral $\int e^{\sinh x} dx$ is nonelementary.

Proof: On putting sinhx=z, we get

$$\int e^{\sinh x} dx = \int \frac{e^z dz}{\sqrt{1+z^2}}$$

which is nonelementary, proved in section 2.7 for φ(x)=x+b, sub-case-I.

Example 6: Show that the integral $\int e^{\tanh x} dx$ is nonelementary.

Proof: On putting tanhx=z, we get

$$\int e^{\tanh x} dx = \int \frac{e^z dz}{(1-z^2)}$$

Which is nonelementary, proved in section 2.1, sub-case-I.

III. Conclusion

From case-I for $f(x)=-x^2$ we get the Gaussian integral, whereas for $f(x)=ix^2$ we get the sine and cosine integrals.

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