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Proof of Third and Sixth Standard Forms of Nonelementary Functions

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Abstract— In this paper we have proved two of the six standard forms of nonelementary functions with examples introduced by Yadav & Sen as indefinite nonintegrable functions. The sine and cosine integrals are the particular case of both the third and sixth forms. Their proofs are based on the strong Liouville's theorem, its special cases and some properties due to Marchisotto & Zakeri. AMS Subject Classification 2010: 97I50, 26A09

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I. Introduction

The natural problem in calculus "*what type of functions cannot be integrated?*" is best answered by Liouville's results. The first example on nonelementary function is the *elliptic integral* due to *John Wallis* [2] in 1655. Such integrals cannot be evaluated in terms of the elementary functions was proved by *Joseph Liouville* [1, 2, 3] in 1833. The main results on functions with nonelementary integrals began with *Liouville's* results specially the strong Liouville theorem and its special cases [2]. *Marchisotto and Zakeri* [2] mentioned two examples 4 and 5 obtained from the special case of strong Liouville theorem. On the basis of the above results *Yadav and Sen* [4, 5] have introduced six standard forms of nonelementary functions from which many nonelementary functions can be derived. In this paper the proofs of the third and sixth forms have been discussed.

II. Third Standard Form Of Nonelementary Functions

Yadav and Sen [4, 5] have stated the third standard form of nonelementary functions as:

"An indefinite integral of the form $\int \frac{f(x)}{g(x)}dx$; where f(x) is a trigonometric (not inverse trigonometric) function, or a hyperbolic (not inverse hyperbolic) function, and g(x) is a polynomial of degree greater than or equal to 1, is always nonelementary".

Proof: We will prove it in three different cases:

Case-I: **When f(x) is a trigonometric (not inverse trigonometric) function and φ(x) be a polynomial of degree ≥1.** Then

2.1. For **f(x)=sinφ(x)**, we have

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\sin\varphi(x)}{g(x)}dx = \frac{1}{2i}\int \frac{e^{i\phi(x)}}{g(x)}dx - \frac{1}{2i}\int \frac{e^{-i\phi(x)}}{g(x)}dx \qquad (2.1.1)$$

By strong Liouville theorem, $\int \frac{e^{i\varphi(x)}}{g(x)}dx$ is elementary if and only if there exists a rational function R(x) which satisfies an identity

$$\frac{1}{g(x)} = R'(x) + i\varphi'(x)R(x) \Rightarrow R'(x) = \frac{1}{g(x)} \text{ and } R(x)=0.$$

But when R(x)=0, R'(x) cannot be $\frac{1}{g(x)}$. Thus no such R(x) exist i.e., it is nonelementary. Similarly we can prove that the integral $\int \frac{e^{-i\varphi(x)}}{g(x)}dx$ is nonelementary. Therefore the given integral is nonelementary.

2.2. For **f(x)=cosφ(x)**, we have

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\cos\varphi(x)}{g(x)}dx = \frac{1}{2}\int \frac{e^{i\phi(x)}}{g(x)}dx + \frac{1}{2}\int \frac{e^{-i\phi(x)}}{g(x)}dx$$

Both are nonelementary proved in section 2.1. Therefore the given integral is nonelementary.

2.3. For **f(x)=tanφ(x)**, we have on putting sexφ(x)=x

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\tan\varphi(x)}{g(x)}dx = \int \frac{\sec\varphi(x)\tan\varphi(x)\varphi'(x)}{\varphi'(x)g(x)\sec\varphi(x)}dx = \int \frac{dz}{zg(x)\varphi'(x)} \quad (2.3.1)$$

Sub-case I: When φ(x)=x+b, then from (2.3.1) we have

$$\int \frac{dz}{zg(x)\varphi'(x)} = \int \frac{dz}{zg(\sec^{-1}z-b)} = \int \frac{\sqrt{z^2-1}dz}{z\sqrt{z^2-1}g(\sec^{-1}z-b)} \quad (2.3.2)$$

For g(x)=x+a, we have from (2.3.2)

$$\int \frac{\tan\phi(x)}{g(x)}dx = \int \frac{\sqrt{z^2-1}dz}{z\sqrt{z^2-1}(\sec^{-1}z+k)}, k = a-b$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{(\sec^{-1}z+k)z} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, hence the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.3.1) we have

$$\int \frac{\tan\phi(x)}{g(x)}dx = \int \frac{\sqrt{z^2-1}dz}{2z\sqrt{z^2-1}\sqrt{\sec^{-1}z+k}\,g\left\{\sqrt{\sec^{-1}z+k}-B\right\}}, k = \frac{b^2-4c}{4}, B = \frac{b}{2} \quad (2.3.3)$$

For g(x)=x+a, (2.3.3) becomes

$$= \int \frac{\sqrt{z^2-1}dz}{2z\sqrt{z^2-1}\sqrt{\sec^{-1}z+k}\left\{\sqrt{\sec^{-1}z+k}+A\right\}}, A = a-B$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{2z\sqrt{\sec^{-1}z+k}\left\{\sqrt{\sec^{-1}z+k}+A\right\}} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

2.4. For **f(x)=cotφ(x)**, we have cosecφ(x)=z

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\cot\phi(x)}{g(x)}dx = \int \frac{\operatorname{cosec}\phi(x)\cot\varphi(x)\phi'(x)}{g(x)\operatorname{cosec}\phi(x)\phi'(x)}dx = \int \frac{-dz}{g(x)\varphi'(x)z} \quad (2.4.1)$$

Sub-case-I: When φ(x)=x+b, then we have from (2.4.1) for g(x)=x+a, letting k=a-b

$$\int \frac{-dz}{g(x)\varphi'(x)z} = \int \frac{-dz}{zg(\operatorname{cosec}^{-1}z-b)} = \int \frac{-\sqrt{z^2-1}dz}{z\sqrt{z^2-1}(\operatorname{cosec}^{-1}z+k)}$$

which can now be proved nonelementary as has been done in section 2.3.

Sub-case-II: When φ(x)=x^2+bx+c, then for g(x)=x+a, we have from (2.4.1)

$$\int \frac{-dz}{g(x)\varphi'(x)z} = \int \frac{-\sqrt{z^2-1}dz}{2(\sqrt{\cos ec^{-1}z+k}-B)z\sqrt{z^2-1}\sqrt{\cos ec^{-1}z+k}}$$

which can now be proved nonelementary as has been done in section 2.3.

2.5. For **f(x)=secφ(x)**, we have on putting $e^{i\varphi(x)}=z$

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\sec\varphi(x)}{g(x)}dx = \int \frac{2dx}{g(x)\left[e^{i\varphi(x)}+e^{-i\varphi(x)}\right]} = \frac{2}{i}\int \frac{dz}{g(x)\varphi'(x)(z^2+1)} \quad (2.5.1)$$

Sub-case-I: When φ(x)=x+b, then from (2.5.1) we have for g(x)=x+a

$$\frac{2}{i}\int \frac{dz}{g(x)\varphi'(x)(z^2+1)} = 2\int \frac{dz}{(\log z+k)(z^2+1)}, i(a-b)=k$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{(\log z+k)(z^2+1)} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.5.1) we have for g(x)=x+a

$$\frac{2}{i}\int \frac{dz}{g(x)\varphi'(x)(z^2+1)} = \int \frac{dz}{i(\sqrt{\log z+k}+M)(z^2+1)\sqrt{\log z+k}}, M = a-\frac{b}{2}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{i(\sqrt{\log z+k}+M)(z^2+1)\sqrt{\log z+k}} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

2.6. For **f(x)=cosecφ(x)**, we have on putting $e^{i\varphi(x)}=z$

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\cos ec\,\varphi(x)}{g(x)}dx = \int \frac{2idx}{g(x)\left[e^{i\varphi(x)}-e^{-i\varphi(x)}\right]} = 2\int \frac{dz}{g(x)\varphi'(x)(z^2-1)} \quad (2.6.1)$$

Sub-case-I: When φ(x)=x+b, then we have from (2.6.1) for g(x)=x+a

$$\int \frac{2dz}{g(x)\varphi'(x)(z^2-1)} = \int \frac{2dz}{g(x)(z^2-1)} = \int \frac{2zdz}{(\cos ec^{-1}z+k)z\sqrt{z^2-1}\sqrt{z^2-1}}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{2}{(\cos ec^{-1}z+k)(z^2-1)} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then we have from (2.6.1) for g(x)=x+a

$$\int \frac{2dz}{g(x)\varphi'(x)(z^2-1)} = \int \frac{dz}{g(x)(z^2-1)\sqrt{\cos ec^{-1}z+k}}, k = \frac{b^2-4c}{4}$$

$$= \int \frac{dz}{(z^2-1)(\sqrt{\cos ec^{-1}z+k}+M)\sqrt{\cos ec^{-1}z+k}}, M = a-\frac{b}{2}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{1}{(z^2-1)(\sqrt{\cos ec^{-1}z+k}+M)\sqrt{\cos ec^{-1}z+k}} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

Case-II: **When f(x) is a hyperbolic (not inverse hyperbolic) function and φ(x) be an arbitrary polynomial of degree ≥1.** Then

2.7. For **f(x)=sinhφ(x)**, we have

$$\int \frac{f(x)}{g(x)}dx = \frac{1}{2}\int \frac{e^{\phi(x)}}{g(x)}dx - \frac{1}{2}\int \frac{e^{-\phi(x)}}{g(x)}dx \qquad (2.7.1)$$

By strong Liouville theorem, the first integral in (2.7.1) $\int \frac{e^{\varphi(x)}}{g(x)}dx$ is elementary if and only if there exists a rational function R(x) satisfying an identity

$$\frac{1}{g(x)} = R'(x) + \varphi'(x)R(x) \qquad (2.7.2)$$

Let R(x)=p(x)/q(x); gcd(p(x), q(x))=1, then from (2.7.2) we get

$$[g(x)p'(x) - q(x) + \varphi'(x)p(x)g(x)] = \frac{p(x)g(x)q'(x)}{q(x)} \qquad (2.7.3)$$

Since gcd(p(x), q(x))=1, from (2.7.3) we have either q(x) divides q'(x) or q(x) divides g(x).

Sub-case-I: When q(x)|q'(x), it means q(x) is a constant. Then without loss of generality we may take R(x)=p(x). Comparing the degrees of x on both side of (2.7.3) results out in a contradiction. Hence q(x) cannot divide q'(x).

Sub-case-II: When q(x)|g(x), let g(x)=ξ(x)q(x), where degree of ξ(x)<degree of g(x). Then from (2.7.3), we have

$$\xi(x)p'(x) - 1 + \varphi'(x)p(x)\xi(x) = \frac{p(x)\xi(x)q'(x)}{q(x)} \qquad (2.7.4)$$

But q(x) does not divide p(x) and q'(x), then it must divide ξ(x). If q(x)|ξ(x), let ξ(x)=q(x)r(x). Then from (2.7.4) we have

$$r(x)[q(x)p'(x) + \varphi'(x)p(x)q(x) - p(x)q'(x)] = 1$$

Which results out in a contradiction by comparing the degrees of x in both sides i.e., such R(x) does not exist. Hence the given function is nonelementary. Similarly we can prove that the second integral in (2.7.1) is nonelementary.

2.8. For **f(x)=coshφ(x)**, we have

$$\int \frac{f(x)}{g(x)}dx = \frac{1}{2}\int \frac{e^{\phi(x)}}{g(x)}dx + \frac{1}{2}\int \frac{e^{-\phi(x)}}{g(x)}dx$$

Both are nonelementary, proved in section 2.7.

2.9. For **f(x)=tanhφ(x)**, we have on putting sechφ(x)=z

$$\int \frac{f(x)}{g(x)}dx = \int \frac{\tanh\varphi(x)}{g(x)}dx = \int \frac{\sec h\varphi(x).\tanh\varphi(x).\varphi'(x)dx}{g(x)\varphi'(x)\sec h\varphi(x)} = \int \frac{dz}{g(x)\varphi'(x)z} \qquad (2.9.1)$$

Sub-case-I: When φ(x)=x+b, then from (2.9.1) we have for g(x)=x+a

$$\int \frac{dz}{g(x)\varphi'(x)z} = \int \frac{dz}{zg(x)} = \int \frac{\sqrt{1-z^2}dz}{z\sqrt{1-z^2}(\sec h^{-1}z + M)}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{1}{z(\sec h^{-1}z + M)} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist, which satisfies this identity. Therefore the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.9.1) we have for g(x)=x+a

$$\int \frac{dz}{g(x)\varphi'(x)z} = \int \frac{\sqrt{1-z^2}dz}{2z\sqrt{1-z^2}(\sqrt{\sec h^{-1}z + k} + B)\sqrt{\sec h^{-1}z + k}}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{1}{2z\sqrt{\sec h^{-1}z+k}(\sqrt{\sec h^{-1}z+k}+B)}=U_0+\sum_{i=1}^{n}C_i\frac{U_i^{'}}{U_i}$$

But no such U_j exist, which satisfies this identity. Therefore the given function is nonelementary.

2.10. For **f(x)=cothφ(x)**, we have on putting cosechφ(x)=z

$$\int\frac{f(x)}{g(x)}dx=-\int\frac{-\operatorname{cosec}h\varphi(x).\coth\varphi(x).\varphi'(x)dx}{g(x)\varphi'(x)\operatorname{cosec}h\varphi(x)}=-\int\frac{dz}{g(x)\varphi'(x)z} \qquad (2.10.1)$$

Sub-case-I: When φ(x)=x+b, then from (2.10.1) we have for g(x)=x+a

$$\int\frac{-dz}{g(x)\varphi'(x)z}=\int\frac{-dz}{zg(x)}=\int\frac{-\sqrt{z^2+1}dz}{z\sqrt{z^2+1}(\operatorname{cosech}^{-1}z+B)}$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{-1}{z(\operatorname{cosech}^{-1}z+B)}=U_0+\sum_{i=1}^{n}C_i\frac{U_i^{'}}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.10.1) we have for g(x)=x+a

$$\int\frac{-dz}{g(x)\varphi'(x)z}=\int\frac{-dz}{2z\sqrt{\operatorname{cosech}^{-1}z+k}(\sqrt{\operatorname{cosech}^{-1}z+k}+B)}$$

which can now be proved nonelementary as has been proved in sub-case-I above.

2.11. For **f(x)=sechφ(x)**, we have on putting $e^{\varphi(x)}$=z

$$\int\frac{f(x)}{g(x)}dx=\int\frac{\sec h\varphi(x)}{g(x)}dx=\int\frac{2dx}{g(x)\left[e^{\varphi(x)}+e^{-\varphi(x)}\right]}=2\int\frac{dz}{g(x)\varphi'(x)(z^2+1)} \qquad (2.11.1)$$

Sub-case-I: When φ(x)=x+b, then from (2.11.1) we have for g(x)=x+a

$$\int\frac{2dz}{g(x)\varphi'(x)(z^2+1)}=\int\frac{2dz}{g(x)(z^2+1)}=\int\frac{2dz}{(\log z+B)(z^2+1)},B=a-b$$

By strong Liouville theorem it is elementary if and only if there exists an identity

$$\frac{2}{(z^2+1)(\log z+B)}=U_0+\sum_{i=1}^{n}C_i\frac{U_i^{'}}{U_i}$$

But no such U_j exist, therefore the given function is nonelementary.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.11.1) we have for(x)=x+a

$$\int\frac{2dz}{g(x)\varphi'(x)(z^2+1)}=\int\frac{2dz}{(z^2+1)\sqrt{\log z+k}(\sqrt{\log z+k}+B)}$$

Which is nonelementary, proved in section 2.5, sub-case-II.

2.12. For **f(x)=cosechφ(x)**, we have on putting $e^{\varphi(x)}$=z

$$\int\frac{f(x)}{g(x)}dx=\int\frac{\operatorname{cosec}h\varphi(x)}{g(x)}dx=\int\frac{2dx}{g(x)\left[e^{\varphi(x)}-e^{-\varphi(x)}\right]}=2\int\frac{dz}{g(x)\varphi'(x)(z^2-1)} \qquad (2.12.1)$$

Sub-case-I: When φ(x)=x+b, then from (2.12.1) we have for g(x)=x+a

$$\int\frac{2dz}{g(x)\varphi'(x)(z^2-1)}=\int\frac{2dz}{g(x)(z^2-1)}=\int\frac{2dz}{(\log z+B)(z^2-1)},(B=a-b)$$

which can now be proved nonelementary as has been done in sections 2.5 and 2.11.

Sub-case-II: When φ(x)=x^2+bx+c, then from (2.12.1) we have on putting g(x)=x+a

$$\int \frac{2dz}{g(x)\varphi'(x)(z^2-1)} = \int \frac{2dz}{(z^2-1)\sqrt{\log z + k}(\sqrt{\log z + k} + B)}$$

Which can be proved nonelementary as has been done in section 2.5, sub-case-II.

We can extend the proofs for higher degree polynomials φ(x) and g(x).

Let us consider some examples based on the third standard form of nonelementary functions:

Example 2.1: Show that the integral $\int \frac{\sin x}{x} dx$ is nonelementary.

Proof: It is nonelementary from example 12 [2, p.302].

Alternate Proof: We have using Euler's identity

$$\int \frac{\sin x}{x} dx = \text{img}\left[\int \frac{e^{ix}}{x} dx\right]$$

Taking g(x)=ix, f(x)=1/x, and applying strong Liouville theorem (special case), it is elementary if and only if there exists a rational function R(x) which satisfies the identity

$$R'(x) + iR(x) = \frac{1}{x} \Rightarrow R'(x) = \frac{1}{x} \;\&\; R(x) = 0$$

which is impossible. Hence the given function is nonelementary.

Example 2.2: Show that the integral $\int \frac{\cosh x}{x} dx$ is nonelementary.

Proof: We have

$$\int \frac{\cosh x}{x} dx = \frac{1}{2}\int \frac{e^x}{x} dx + \frac{1}{2}\int \frac{e^{-x}}{x} dx$$

Both are well defined and well proved nonelementary functions from example 2 [2].

Example 2.3: Show that the integral $\int \frac{\sinh x}{(ax^2+b)} dx, a \neq 0$ is nonelementary.

Proof: We have

$$\int \frac{\sinh x}{(ax^2+b)} dx = \frac{1}{2}\int \frac{e^x}{(ax^2+b)} dx - \frac{1}{2}\int \frac{e^{-x}}{(ax^2+b)} dx$$

Where

$$\int \frac{e^x}{(ax^2+b)} dx = \frac{1}{2iak}\left[\int \frac{e^x}{(x-ik)} dx - \int \frac{e^x}{(x+ik)} dx\right]; k^2 = \frac{b}{a}$$

Both are nonelementary, proved in section 2.3, sub-case-I. Similarly

$$\int \frac{e^{-x}}{(ax^2+b)} dx = \frac{1}{2iak}\left[\int \frac{e^{-x}}{(x-ik)} dx - \int \frac{e^{-x}}{(x+ik)} dx\right]; k^2 = \frac{b}{a}$$

Both are nonelementary, proved in section 2.3, sub-case-I. Therefore the given function is nonelementary.

Example 2.4: Show that the integral $\int \frac{\tan x}{x} dx$ is nonelementary.

Proof: We have on putting secx=z

$$\int \frac{\tan x}{x} dx = \int \frac{\sec x \tan x}{x \sec x} dx = \int \frac{dz}{z \sec^{-1} z} = \int \frac{\sqrt{z^2-1}dz}{z\sqrt{z^2-1}\sec^{-1} z}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{z\sec^{-1}z}=U_0'+\sum_{i=1}^{n}C_i\frac{U_i'}{U_i}$$

But no such U_j exist. Hence the given function is nonelementary.

Example 2.5: Show that the integral $\int\frac{\tan x}{(ax^3+x^2+b)}dx$ is nonelementary.

Proof: We have on putting secx=z

$$\int\frac{\tan x}{(ax^3+x^2+b)}dx=\int\frac{\sec x\tan x}{(ax^3+x^2+b)\sec x}dx=\int\frac{dz}{z\left[a(\sec^{-1}z)^3+(\sec^{-1}z)^2+b\right]}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity

$$\frac{1}{z[a(\sec^{-1}z)^3+(\sec c^{-1}z)+b]}=U_0'+\sum_{i=1}^{n}C_i\frac{U_i'}{U_i}$$

Giving different values of a and b, we can find that no such U_j exist. Hence the given function is nonelementary.

III. Sixth Standard Form Of Nonelementary Functions

Yadav & Sen [4, 5] have stated sixth standard form of nonelementary functions as:

"An indefinite integral of the form $\int\frac{f(x).g(x)}{h(x)}dx$, where f(x), h(x) are polynomials in x (degree of h(x) is greater than the degree of f(x)) and g(x) is a trigonometric (not inverse trigonometric) or a hyperbolic (not inverse hyperbolic) function is always nonelementary."

Proof: There arise two different cases depending on f(x) and h(x):

Case-I: When f(x) divides h(x) exactly, let f(x)/h(x)=1/φ(x), then we have

$$\int\frac{f(x).g(x)}{h(x)}dx=\int\frac{g(x)}{\phi(x)}dx$$

which is third standard form and whose proofs have been discussed above.

Case-II: When f(x) does not divide h(x) exactly. Let us denote f(x)/h(x)=H(x), where H(x) a rational function with gcd(f(x), h(x))=1. If it is not so, we can make it by dividing them by their common factors. Then the given integral can be written as

$$\int\frac{f(x).g(x)}{h(x)}dx=\int H(x)g(x)dx \tag{3.A}$$

Now we consider the following sub-cases:

Sub-case-I: **When g(x) be a trigonometric (not inverse trigonometric) function and φ(x) be a polynomial of degree ≥1.** Then

3.1. For g(x)=sinφ(x), we have from (3.A)

$$\int H(x)g(x)dx=\int H(x)\sin\varphi(x)dx=\frac{1}{2i}\int H(x)e^{i\varphi(x)}dx-\frac{1}{2i}\int H(x)e^{-i\varphi(x)}dx \tag{3.1.1}$$

Now by strong Liouville theorem, $\int H(x)e^{i\varphi(x)}dx$ is elementary if and only if there exists a rational function R(x) which satisfies an identity

$$H(x)=R'(x)+i\varphi'(x)R(x)\ \Rightarrow R'(x)=H(x)$$ and R(x)=0 because φ'(x)≠0.

But when R(x)=0, R'(x) cannot be H(x). Thus no such R(x) exists, so it is nonelementary. Similarly we can prove that the second integral in (3.1.1) is nonelementary. Therefore the given integral (3.A) is nonelementary for g(x)=sinφ(x) for any H(x).

3.2. For g(x)=cosφ(x), we have from (3.A)

$$\int H(x)g(x)dx=\int H(x)\cos\varphi(x)dx=\frac{1}{2i}\int H(x)e^{i\varphi(x)}dx+\frac{1}{2i}\int H(x)e^{-i\varphi(x)}dx \tag{3.2.1}$$

which can now be proved nonelementary as has been done in section 3.1.

3.3 For g(x)=tanφ(x), we have from (3.A) on putting secφ(x)=z

$$\int H(x)g(x)dx = \int H(x)\tan\varphi(x)dx = \int \frac{H(x)\sec\varphi(x)\tan\varphi(x)\varphi'(x)}{\sec\varphi(x)\varphi'(x)}dx = \int \frac{H(x)dz}{z\varphi'(x)} \quad (3.3.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.3.1) we have

$$\int \frac{H(x)dz}{z\varphi'(x)} = \int \frac{(x+a)dz}{z(x^2+cx+d)} = \int \frac{(\sec^{-1}z+a-b)\sqrt{z^2-1}dz}{z\sqrt{z^2-1}\{(\sec^{-1}z-b)^2+c(\sec^{-1}z-b)+d\}}$$

By strong Liouville theorem it is elementary if and only if there exists an identity of the form

$$\frac{(\sec^{-1}z+a-b)}{z\{(\sec^{-1}z-b)^2+c(\sec^{-1}z-b)+d\}} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exists, hence it is nonelementary.

3.4. For g(x)=cotφ(x), we have from (3.A) on putting cosecφ(x)=z

$$\int H(x)g(x)dx = \int H(x)\cot\varphi(x)dx = \int \frac{H(x)\operatorname{cosec}\varphi(x)\cot\varphi(x)\varphi'(x)}{\operatorname{cosec}\varphi(x)\varphi'(x)}dx = \int \frac{-H(x)dz}{z\varphi'(x)} \quad (3.4.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.4.1) we have

$$\int \frac{-H(x)dz}{z\varphi'(x)} = \int \frac{-(\operatorname{cosec}^{-1}z+a-b)dz}{z\{(\operatorname{cosec}^{-1}z-b)^2+c(\operatorname{cosec}^{-1}z-b)+d\}}$$

which can now be proved nonelementary as has been done in section 3.3.

3.5. For g(x)=secφ(x), we have from (3.A) on putting $e^{i\varphi(x)}$=z

$$\int H(x)g(x)dx = \int H(x)\sec\varphi(x)dx = \int \frac{2H(x)i\varphi'(x)e^{i\varphi(x)}}{i\varphi'(x)\{e^{i2\varphi(x)}+1\}}dx = \int \frac{2H(x)dz}{i\varphi'(x)(z^2+1)} \quad (3.5.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.5.1) we have

$$\int \frac{2H(x)dz}{i\varphi'(x)(z^2+1)} = \int \frac{2(-i\log z+a-b)dz}{i\{(-i\log z-b)^2+c(-i\log z-b)+d\}(z^2+1)}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{2(-i\log z+a-b)}{i\{(-i\log z-b)^2+c(-i\log z-b)+d\}(z^2+1)} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist. Therefore the given function is nonelementary.

3.6. For g(x)=cosecφ(x), we have from (3.A) on putting $e^{i\varphi(x)}$=z

$$\int H(x)g(x)dx = \int H(x)\operatorname{cosec}\varphi(x)dx = \int \frac{2H(x)i\varphi'(x)e^{i\varphi(x)}}{\varphi'(x)\{e^{i2\varphi(x)}-1\}}dx = \int \frac{2H(x)dz}{\varphi'(x)(z^2-1)} \quad (3.6.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.6.1) we have

$$\int \frac{2H(x)dz}{\varphi'(x)(z^2-1)} = \int \frac{2(-i\log z+a-b)dz}{\{(-i\log z-b)^2+c(-i\log z-b)+d\}(z^2-1)}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{2(-i\log z+a-b)}{\{(-i\log z-b)^2+c(-i\log z-b)+d\}(z^2-1)} = U_0 + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

But no such U_j exist. Therefore the given function is nonelementary.

Case-II: **When g(x) is a hyperbolic (not inverse hyperbolic) function and φ(x) be a polynomial of degree ≥1.** Then

3.7. For g(x)=sinhφ(x), we have from (3.A)

$$\int H(x)g(x)dx = \int H(x)\sinh\varphi(x)dx = \frac{1}{2}\int H(x)e^{\varphi(x)}dx - \frac{1}{2}\int H(x)e^{-\varphi(x)}dx \quad (3.7.1)$$

Now by strong Liouville theorem $\int H(x)e^{\varphi(x)}dx$ is elementary if and only if there exists a rational function R(x) which satisfies an identity of the form

$$H(x) = R'(x) + \varphi'(x)R(x) \Rightarrow \frac{f(x)}{h(x)} = R'(x) + \varphi'(x)R(x)$$

Let $R(x)=\frac{p(x)}{q(x)}$, where $\gcd\{p(x),q(x)\}=1$, then we have

$$h(x)p'(x)-f(x)q(x)+\varphi'(x)h(x)p(x)=\frac{h(x)p(x)q'(x)}{q(x)}$$

Which implies that q(x) | h(x). Let h(x)=q(x).r(x), then we have

$$r(x)p'(x)-f(x)+\varphi'(x)p(x)r(x)=\frac{r(x)p(x)q'(x)}{q(x)}$$

Which implies that q(x) | r(x). Let r(x)=q(x).ξ(x), then we have

$$q(x)p'(x)-p(x)q'(x)+\varphi'(x)p(x)q(x)=\frac{f(x)}{\xi(x)}$$

Which implies that ξ(x) | f(x). Let f(x)=ξ(x).ψ(x). Then we have

$$q(x)p'(x)-p(x)q'(x)+\varphi'(x)p(x)q(x)=\psi(x) \quad (3.7.2)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.7.2) we have, since ξ(x)|f(x)=x+a, which implies that either ξ(x)=k or k(x+a) and f(x)=x+a=ξ(x).ψ(x) implies that either ψ(x)=M or M(x+a), where both k and M are constants. But for ψ(x)=M or M(x+a), we get a contradiction from (3.7.2) by comparing the degrees of x in both sides, i. e., such R(x) does not exist. Hence it is nonelementary for φ(x)=x+b, f(x)=x+a and h(x)=x^2+cx+d. In the similar way we can prove that the second integral $\int H(x)e^{-\varphi(x)}dx$ is nonelementary. Therefore the given function is nonelementary.

3.8. For g(x)=coshφ(x), we have from (3.A)

$$\int H(x)g(x)dx=\int H(x)\cosh\varphi(x)dx=\frac{1}{2}\int H(x)e^{\varphi(x)}dx+\frac{1}{2}\int H(x)e^{-\varphi(x)}dx$$

which can now be proved nonelementary as has been done in section 3.7.

3.9. For g(x)=tanhφ(x), we have from (3.A) on putting sechφ(x)=z

$$\int H(x)g(x)dx=\int H(x)\tanh\varphi(x)dx=\int\frac{H(x)\operatorname{sech}\varphi(x)\tanh\varphi(x)\varphi'(x)}{\operatorname{sech}\varphi(x)\varphi'(x)}dx=\int\frac{-H(x)dz}{z\varphi'(x)} \quad (3.9.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.9.1) we have

$$\int\frac{-H(x)dz}{z\varphi'(x)}=\int\frac{-(x+a)dz}{z(x^2+cx+d)}=\int\frac{-(\operatorname{sech}^{-1}z+a-b)\sqrt{1-z^2}\,dz}{z\sqrt{1-z^2}\{(\operatorname{sech}^{-1}z-b)^2+c(\operatorname{sech}^{-1}z-b)+d\}}$$

which can now be proved nonelementary as has been done in section 3.4.

3.10. For g(x)=cothφ(x), we have from (3.A) on putting cosechφ(x)=z

$$\int H(x)g(x)dx=\int H(x)\coth\varphi(x)dx=\int\frac{H(x)\operatorname{cosech}\varphi(x)\coth\varphi(x)\varphi'(x)}{\operatorname{cosech}\varphi(x)\varphi'(x)}dx=\int\frac{-H(x)dz}{z\varphi'(x)} \quad (3.10.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.10.1) we have

$$\int\frac{-H(x)dz}{z\varphi'(x)}=\int\frac{-(\operatorname{cosech}^{-1}z+a-b)\sqrt{z^2+1}\,dz}{z\sqrt{z^2+1}\{(\operatorname{cosech}^{-1}z-b)^2+c(\operatorname{cosech}^{-1}z-b)+d\}}$$

which can now be proved nonelementary as has been done in section 3.4.

3.11. For g(x)=sechφ(x), we have from (3.A) on putting $e^{\varphi(x)}$=z

$$\int H(x)g(x)dx=\int H(x)\operatorname{sech}\varphi(x)dx=\int\frac{2H(x)\varphi'(x)e^{\varphi(x)}}{\varphi'(x)\{e^{2\varphi(x)}+1\}}dx=\int\frac{2H(x)dz}{\varphi'(x)(z^2+1)} \quad (3.11.1)$$

Let us take φ(x)=x+b, f(x)=x+a, h(x)=x^2+cx+d. Then from (3.11.1) we have

$$\int\frac{2H(x)dz}{\varphi'(x)(z^2+1)}=\int\frac{2(\log z+a-b)dz}{\{(\log z-b)^2+c(\log z-b)+d\}(z^2+1)}$$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{2(\log z+a-b)}{\{(\log z-b)^2+c(\log z-b)+d\}(z^2+1)}=U_0+\sum_{i=1}^{n}C_i\frac{U_i'}{U_i}$$

But no such U_j exist. Therefore the given function is nonelementary.

3.12. For g(x)=cosechφ(x), we have from (3.A) on putting $e^{\varphi(x)}=z$

$$\int H(x)g(x)dx = \int H(x)\operatorname{cosech}\varphi(x)dx = \int \frac{2H(x)\varphi'(x)e^{\varphi(x)}}{\varphi'(x)\{e^{2\varphi(x)}-1\}}dx = \int \frac{2H(x)dz}{\varphi'(x)(z^2-1)} \quad (3.12.1)$$

Let us take φ(x)=x+b, f(x)=x+a, $h(x)=x^2+cx+d$. Then from (3.12.1) we have

$$\int \frac{2H(x)dz}{\varphi'(x)(z^2-1)} = \int \frac{2(\log z + a - b)dz}{\{(\log z - b)^2 + c(\log z - b) + d\}(z^2-1)}$$

which can now be proved nonelementary as has been done in section 3.11.

Similarly we can extend the proofs for higher degree polynomials f(x), h(x) and φ(x).

Let us consider some examples on sixth standard form of nonelementary functions:

Example 3.1: Show that the integral $\int \frac{(x^2+2).\tan x}{(3x^4+4x)}dx$ is nonelementary.

Proof: We have on putting secx=z $\int \frac{(x^2+2).\tan x}{(3x^4+4x)}dx = \int \frac{\{(\sec^{-1}z)^2+2\}\sqrt{z^2-1}}{\{3(\sec^{-1}z)^4+4(\sec^{-1}z)\}z\sqrt{z^2-1}}dz$

By strong Liouville theorem, it is elementary if and only if there exists an identity of the form

$$\frac{\{(\sec^{-1}z)^2+2\}}{\{3(\sec^{-1}z)^4+4(\sec^{-1}z)\}z} = U_0' + \sum_{i=1}^{n} C_i \frac{U_i'}{U_i}$$

Considering different possible forms of U_j like $\log\{3(\sec^{-1}z)^4+4(\sec^{-1}z)\}$, we find that no such U_j exist. Therefore the given function is nonelementary.

Example 3.2: Show that the integral $\int \frac{(2x^3+3x).\sin(3x^2+2)}{(2x^4+6x)}dx$ is nonelementary.

Proof: Taking

$$\frac{(2x^3+3x)}{(2x^4+6x)} = f(x), \quad 3x^2+2 = g(x)$$

We have

$$\int \frac{(2x^3+3x).\sin(3x^2+2)}{(2x^4+6x)}dx = \int f(x)\sin g(x)dx = \frac{1}{2i}\left[\int f(x)e^{ig(x)}dx - \int f(x)e^{-ig(x)}dx\right]$$

Now from strong Liouville theorem (special case), $\int f(x)e^{ig(x)}dx$ is elementary if and only if there exists a rational function R(x) which satisfies an identity of the form

$$f(x) = R'(x) + ig'(x)R(x) \quad (B)$$

$$\Rightarrow \frac{2x^3+3x}{2x^4+6x} = R'(x) + i6xR(x) \Rightarrow 6xR(x) = 0 \Rightarrow R(x) = 0.$$

But R(x) cannot be zero, i. e., no such R(x) exist, which satisfies the identity (B). This implies that $\int f(x)e^{ig(x)}dx$ is nonelementary. Similarly we can show that the second integral $\int f(x)e^{-ig(x)}dx$ is nonelementary. Therefore the given integral is nonelementary.

IV. Conclusion

For φ(x)=x, g(x)=x in sections 2.1 and 2.2 as well as for f(x)=1, h(x)=x, g(x)=sinx, cosx in sections 3.1 and 3.2, we get the sine and cosine integrals respectively

$$\int \frac{\sin x}{x}dx \text{ and } \int \frac{\cos x}{x}dx$$

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REFERENCES

[1] G. H. Hardy, The Integration of Functions of a Single Variable, 2nd Ed., Cambridge University Press, London, (Reprint 1928), 1916

[2] E. A. Marchisotto & G. A. Zakeri, An Invitation to Integration in Finite Terms, The College Mathematics Journal, Mathematical Association of America, 25:4 (1994) 295-308.

[3] J. F. Ritt, Integration in Finite Terms: Liouville's Theory of Elementary Methods, Columbia University Press, New York, 1948

[4] D. K. Yadav & D. K. Sen, Revised paper on Indefinite Nonintegrable Functions, Acta Ciencia Indica, 34:3 (2008) 1383-1384.

[5] D. K. Yadav, A Study of Indefinite Nonintegrable Functions, Ph. D. Thesis, Vinoba Bhave University, Hazaribag, Jharkhand, 2012.