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Combinatorial Proof of Cayley-Hamilton Theorem

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Abstract: The aim of the paper is to prove the Cayley – Hamilton Theorem using a combinatorial derivation of the determinant of a matrix in conjuction i_{th} elementary graph theory. First , introduce notation for a particular partial permutation. Then divide the marix's characteristics polynomial into two groups composed of positive and negative terms.Then concludes by asserting that these to groups are equal and opposite.

Keywords: Cayley- Hamilton Theorem, Partial Permutation, Matrices, Characteristics Polynomial.

I. INTRODUCTION

Defintion 1.1. If A is an n×n matrix ,then the characteristic polynomial of A is defined to be $P_A(x)$= det(xI-A). This is a polynomial in x of degree n with leading term x^n . the constant term c_0 of a polynomial q(x) is interpreted as c_0 I in q(A).
Theorem 1.2 (Cayley – Hamilton Theorem). If A is an n×n matrix ,then $p_A(A)=0$, the zero matrix.
Theorem 1.3 If q≠0 is a quaternion of the form q-= a+bi+cj+dk with a,b,c,d, being real , then $q^2 - 2aq + (a^2+b^2+c^2+d^2)=0$

$$q^{-1} = \frac{\bar{q}}{|q|}$$

$$= \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$$

$$= \frac{2a}{a^2+b^2+c^2+d^2} - \frac{a+bi+cj+dk}{a^2+b^2+c^2+d^2}$$

$$= \frac{1}{a^2+b^2+c^2+d^2}(2a-q)$$

$$\Rightarrow a^2+b^2+c^2+d^2 = 2aq - q2$$

$$\Rightarrow q2 - 2aq + (a^2+b^2+c^2+d^2) = 0$$

If one represents a quaternion q= a+bi+cj+dk as a matrix,

$$A = \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix},$$

$P_A(A) = A^2 - 2aA + (a^2+b^2+c^2+d^2)I = 0$, and the polynomial given in Theorem 1.3 is characteristic polynomial of A

II. GENERALIZATION OF CAYLEY HAMILTON THEOREM

Theorem 2.1 (Cayley-Hamilton Theorem). For any n × n Matrix A, $P_A(A)=0$.
Proof. Let D(x) be the matrix with polynomial entries $D(x)= adj(xI_n-A)$, So $D(x)(xI-A)= det(xI_n-A)I_n$. Since each entry in D(x) is the determinant of an *(n-1) X((n-1)* submatrix of (xI_n-A), each entry of *D(x)* is a polynomial of degree less than or equal to n-1. It folowws that there exist matrices $D_0, D_1,\ldots D_{n-1}$ with entries from C such that $D(x) = D_{n-1} x^{n-1}+\ldots+ D_1 x + D_o$. Then the matrix equation follows

$$det(xI_n-A) I_n = (x I_n-A)\, adj(xI_n-A) = (xI_n-A)D(x)$$

Substituting $p_A(x)= det(xI_n-A)$, (and using the fact that scalars commute with matrix)

$$X^nI_n+ b_{n-1}x^{n-1}I_n+\ldots\ldots+ b_1xI_n+b_0I_n$$
$$= p_A(x)\, I_n= det(xI_n-A)I_n$$
$$=(xI_n-A)adj(xI_n-A)$$
$$=(xI_n-A)(x^{n-1}D_{n-1}+\ldots+xD_1+D_o)$$
$$=x^{nD}{}_{n-1}- x^{n-1}AD_{n-1}+ x^{n-1}D_{n-2}-x^{n-2}AD_{n-2}+\ldots+ xD_0-AD_o$$
$$=x^{nD}{}_{n-1}+ x^{n-1}(-AD_{n-1}+ D_{n-2}) +\ldots+(-AD_1+D_o) -AD_o$$

Since two polynomials are equal if and only if their coefficients are equal , the coefficient matrices are equal ; that is , $I_n=D_{n-1}$, $b_{n-1}I_n= (-AD_{n-1}+D_{n-2}),\ldots,b_1I_n=-(AD_1+D_0)$, *and* $b_0I_n=-AD_0$. This means that A may be substituted for the variable x in the equation (2.1) to conclude

$P_A(A) = A^n + b_{n-1} A^{n-1} + \ldots\ldots + b_1 A + b_0 I_n$

$= A^n D_{n-1} + A^{n-1}(-AD_{n-1} + D_{n-2}) + \ldots\ldots + A(-AD_1 + D_0) - AD_0$

$= A^n D_{n-1} - A^n D_{n-1} + A^{n-1} D_{n-2} - A^{n-1} D_{n-2} + \ldots\ldots\ldots. AD_0 - AD_0$

$= 0$

This proves the theorem

III. COMBINATORIAL PROOF OF CAYLEY- HAMILTON THEOREM

3.1.1. *Partial permutation σ*

A partial permutation of {1,…,n}is a bijection σ of a subset of {1,…,n} onto itself. The domain of σ is denoted by dom σ.The cardinality of dom σ is called the degree of σ and is denoted by$|\sigma|$.

A complete permutation whose domain is {1,….,n}. If σ is a partial permutaion of {1,...,n} , then the completion of σ , denoted $\hat{\sigma}$, is the complete permutation of {1,...,n} defined by

*Defintion 3.1.***1**. $\hat{\sigma}(i) = \begin{Bmatrix} \sigma(i) & if\ i \in dom\ \sigma \\ i & if\ i \in \{1, \ldots, n\} \backslash dom\sigma \end{Bmatrix}$

Definition 3.1.2. The signature of a complete permutation $\hat{\sigma}$ denoted sgn($\hat{\sigma}$), is +1if the total number of inversion in $\hat{\sigma}$ is even and -1 if that number is odd.

Definition 3.1.3. The signature of a complete permutation σ , denoted sgn(σ), is defined by sgn(σ)=$(-1)^{|\sigma|} sgn(\hat{\sigma})$.

The characteristics polynomial , $p_A(x)$, of a matrix is the sum o certain products of elements of that matrix and poers of x . It is shonbelow that the pairs of indices (i,j) appearing in one of these products an be described using partial permutations. There is a relation between the elements of a given product. Namely , their subscripts are ordered pairs (i, σ(i)) where σ is a partial permutation of {1,…n} and $if\ i \in dom\ \sigma$. For example , if $A \in M_3$, the terms of $p_A(x)$ are:

$\lvert\sigma\rvert = 0$	$\lvert\sigma\rvert = 1$	$\lvert\sigma\rvert = 2$	$\lvert\sigma\rvert = 3$
x^3	$-a_{11}x^2$	$a_{11}a_{22}\ x$	$-a_{11}a_{22}\ a_{33}$
	$-a_{22}x^2$	$a_{11}a_{33}\ x$	$a_{11}a_{23}\ a_{32}$
	$-a_{33}x^2$	$a_{22}a_{33}\ x$	$a_{22}a_{13}\ a_{31}$
		$-a_{12}a_{21}\ x$	$a_{33}a_{12}\ a_{21}$
		$-a_{13}a_{31}\ x$	$-a_{12}a_{23}\ a_{31}$
		$-a_{23}a_{32}\ x$	$-a_{13}a_{32}\ a_{21}$

Table 3.1: Terms of $p_A(x)$ of M_3

Note that here the coeffiecient of x is the sum of signed terms whose indices come from the 6 partial permutations of {1,2,3} of order 2:

$a_{11}a_{22}$ corresponds to σ_1=(1)(2),

$a_{12}a_{21}$ corresponds to σ_4=(1,2), etc.

It will be shown in general that each partial permutation of {1,…,n} of order q yields a signed term , which is one of he summands in the coefficient of x^{n-q}. Graph theory will help one visualize the partial permutations involved. Let $dom\ \sigma$ be vertices and the ordered pairs (i, σ(i)) ,where $i \in dom\ \sigma$, be directed edges. The bijective properties of σ mandate that this graph is a graph of adjoined cycles.

This transition from combinatorics to graph theory allows one to use an elegant set an example to prove the Cayley – Hamilton Theorem. To use this transiton $p_A(x)$ must be described in some detail.

3.1.2 Positive and negative parts of the charactristics polynomial

Let A = $(a_{ij}) \in M_n$ overC. As defined , the characteristics polynomial of A is :

$$P_A(x) = \det(xI - A) \sum_{\sigma \in s_n} sgn(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

Where, $b_{ij} = \begin{Bmatrix} -a_{ij} & if\ i \neq j \\ x - a_{ij} & if\ i = j \end{Bmatrix}$

and where the summation is overall complete permutations $\hat{\sigma}$ of {1,…,n}. It follows that the coefficient of x^{n-q} in $p_A(x)$ is

$$\sum_{|\sigma|=q} sgn(\sigma) \prod_{i \in dom\ \sigma} a_{i,\sigma(i)}.$$

To see this, notice that from Equation 3.1 , the coefficient of x^{n-q} comes from the terms $b_{i,n(i)} = b_{ii} = x - a_{ii}$ for n-q of the indices, and $b_{i,n(i)} = -a_{i,n(i)}$ for the other q indices. Such n is completion of a partial permutation σ of order q. The q terms corresponding to the $a_{i,n(i)}$ are called a_{ij} corresponding to σ. For example, if $A \in M_3$ and q=2, then $x^{(n-q)} = x^1$, and σ permutes 2 different elements of {1,2,3} so the coefficients of x^1 are various products of to terms. This analysis gives a precise algorithm for finding the coefficients of a specific variable in the polynomial $p_A(x)$ for $A \in M_n$.

1. Note the variable's degree *n-q*
2. Find all partial permutations σ such that $|\sigma|$=q
3. Create the ordered pairs (i,j)= $(i, \sigma(i)),\ i \in dom\ \sigma$
4. Multiply all $a_{i,\sigma(i)}$ corresponding to a particular σ and attach the approriate sign.
5. The sum of these signed products is the coefficient of x^{n-q}

Referencing Table 3.1 , if n=3, q=2 , and σ is a partial permutation of order 2, then $|\sigma|$=2 and σ creates two ordered pairs or two a_{ij}*'s* . Thus , each term in the column represents the signed product of the two a_{ij}*'s* whichare the domain of each of the 6 σ 's .With the coefficients ofeach variable of p_A(x) properly defined , $p_A(x)$ is the sum of these terms. These coefficients are either positive or negative and the following definitions are created.

Definition 3.1.3. $p_A(x)=p_A^+(x) - p_A^-(x)$,

Where

Definition 3.1.4. $p_A^+(x) = \sum_{q=0}^{n}\left(\sum_{\substack{|\sigma| \\ sgn\sigma = 1}} \prod_{i \in dom\sigma} a_{i,\sigma(i)}\right) x^{n-q}$,

and,

Definition 3.1.5. $p_A^-(x) = \sum_{q=0}^{n}\left(\sum_{\substack{|\sigma| \\ sgn\sigma = -1}} \prod_{i \in dom\sigma} a_{i,\sigma(i)}\right) x^{n-q}$

Note that if $A \in M_3$, the variables of degree 1 qnd 0 have a combination of $p_A^+(x) and\ p_A^-(x)$ terms.

With the desciption of p_A(x), the cayley –Hamilton Theorem may be rewritten as follows:

Main Theorem =$\boldsymbol{p_A^+(x) = p_A^-(x)}$

With the help of some basic graph theory definitions this theorem can be proved.

IV. CHARACTERSTICS

Straubing's combinatorial proof of the cayley – Hamilton Theorem exploits three aspects of p_A(A). First, it elegantly explain the relationship betwwn positvi and negative terms of p_A(A). Second , the proof illuminates the importance of n. It is cornerstone for the entire proof. P_A(A) is a sum of n products of elements of A. Third the proof introduces a cyclic property to p_A(A) are partial permuatins and therefore may be represented by adjoint cycles.

V. APPLICATION OF CAYLEY– HAMILTON THEOREM

To prove the Cayley- Hamilton theorem through combinatorics and graph theory, Straubingactually creates an algorithm for finding the coefficients of p_A(x) is the sum of a function of all partial permutations with certain constraints. Using this algorithm , individual coefficients of p_A(x) may be generated without having to generate all the coefficients and without ever knowing the eigenvalues of A.

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