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Optimal Control Theory Approach to Solve Constrained Production and Inventory System

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Abstract: This paper proposes an optimal control of production inventory system. The production and inventory system is solved using optimal control theory. We present finite continuous time linear optimal control model to control dynamic price and production flow rate to maximize total revenue minus total cost. Where the demand is linear with price, the production flow rate of the product not exceeds the maximum production capacity rate. Total cost consists of the manufacture holding cost, buyer holding cost and production cost. Pontryagin Maximum Principle is applied to find the controllers of the model.

Keywords: Production Inventory System, Optimal Control. Pontryagin Maximum Principle

I-Introduction

Recently the optimal control theory is used to solve problems of inventory and production systems. Researchers are focusing on estimating the effect of the changes in the demand with time in logistics. Continuous time optimal control models provide a convenient way for awareness the behavior of systems where system dynamic plays a significant role. A lot of research is focusing on models considering all features of these problems. In particular the most attractive feature of these models is to provide good scheduling production and inventory polices in diversity of settings. Moreover they approximate well the fundamental stochastic of problems in a deterministic way. A continuous time method has an excellent feature which is not introducing any approximation to the real setting; it provides the accurate solution of the system.

Literature Review: S.P.Sethi, G. L. Thompson, (2000) presented a model in which the factory produces a single homogenous product and has a finished goods store. They assumed that the model is similar to HMMS model. The model is formulated and completely solved using optimal control theory .The steady state solution is obtained when horizon is infinity. The optimization problem is to minimize total cost of the model. M. A. Baten , A. A. Kamil. (2009) presented a model in which the inventory –production system with two parameters Weibull distributed deterioration items in which the inventory model is considered as linear optimal control problem and the model is solved by pontryagin maximum principle, the solution of the optimal control problem is solved analytically. Yinkuan Gu, Hongxia Zhang, 2011, present a model to control supply Chain. The main aim of the model is to make the total cost of the supply chain minimum. Ghodrat Alah Emamverdi, et al (2011) presented optimal control of production inventory system considering deteriorating items. The main objective of this study is to determine minimum total production and inventory costs. D. Ivanov ,B. Sokolov (2013) present a model to analyze and to achieve designed economic performance in a actual time uncertain and disturbance execution environment is vital and modern issue in many supply chains. This study is the first papers addressing the operative perspective of the supply chain dynamic domain.

II- Assumptions and Notations

To establish the model the following notations and assumptions are used. Consider a factory producing a certain product and having homogenous finishing goods warehouse. The factory distributes the product to a buyer which sells some and stores the rest. The objective is to determine the optimal price and optimal production rate to maximize the total profit of both the buyer and the vendor.

Inputs

T : Time horizon

Q : Vendor production capacity

h_b : Buyer holding cost.

h_v : Vendor holding cost.

c : The production cost coefficient

a, b: Coefficients used for the product at time t in linear relationship between price and demand $d = a(t) - b(t)P(t)$

$\hat{u}_v$: Production goal level of the vendor

$\hat{I}_b$: Inventory goal level of the buyer

$\hat{I}_V$: Inventory goal level of the vendor

The meaning of inventory goal level $\hat{I}$ is that a safety stock that the company wants to keep on hand. Similarly the meaning of production goal level $\hat{u}_v$ is explained as most efficient level at which it is required to run the factory.

Outputs

$P(t)$: Price of one unit of the product at time t (control variable)

$u_v(t)$: Production rate at time t (control variable.)

I_b: Inventory level (number of units) of buyer at time t (state variable).

I_v: Inventory level (umber of units) of vendor at time t (state variable).

The problem seeks to maximize the revenue minus the inventory and productions costs. The objective function of the model can be written as:

Maximize

$$\int_0^T \left[p(t)d(t) - \frac{1}{2} h_b \left(I_b(t) - \hat{I}_b\right)^2 - \frac{1}{2} h_v \left(I_v(t) - \hat{I}_v\right)^2 - \frac{c}{2}\left(u_v(t) - \hat{u}_v\right)^2 \right] dt \qquad (1)$$

Subject to

$$\dot{I}_b = -d(t) \qquad (2)$$

$$\dot{I}_v = u_v(t) - d(t) \qquad (3)$$

$$d = a(t) - b(t)P(t) \quad \forall\, t \in [0, T] \qquad (4)$$

$$u_v(t) \le Q \quad \forall\, t \in [0, T] \qquad (5)$$

$$P(t) \le \frac{a(t)}{b(t)} \qquad (6)$$

$$u_v(t),\ I_b(t),\ I_v(t),\ P(t) \ge 0 \qquad (7)$$

With initial condition

$I_b(0) = I_{bo}$, $I_v(0) = I_{vo}$.

III. The Mathematical Model and Analysis

We start by applied Pontryagin maximum principle [6],[12]

Hamiltonian function

$$H = p(t)d(t) - \frac{1}{2} h_b \left(I_b(t) - \hat{I}_b\right)^2 - \frac{1}{2} h_v \left(I_v(t) - \hat{I}_v\right)^2 - \frac{c}{2}\left(u_v(t) - u_v\right)^2 - \lambda_1 d(t) + \lambda_2 (u - d(t)) \qquad (8)$$

Lagrangian function

$$L = H + \mu_i\, g_i + \eta_i\, f_i \qquad (9)$$

Where $\mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2$, satisfy the complementary slackness conditions

$$\mu_1 \ge 0\ ,\ \mu_1 u_v = 0\ ,\ \mu_2 \ge 0\ ,\ \mu_2 (Q - u_v) = 0\ ,\ \mu_3 \ge 0\ ,\ \mu_3 \left(\frac{a}{b} - p(t)\right) = 0\ ,\ \mu_4 \ge 0\ ,\ \mu_4 P(t) = 0\ ,\ \eta_1 \ge 0\ ,$$

$$\eta_1 I_b = 0\ ,\ \dot{\eta}_1 \le 0\ ,\ \eta_2 \ge 0\ ,\ \eta_2 I_v = 0\ ,\ \dot{\eta}_2 \le 0 \qquad (10)$$

From equation (7) and equation (10) it is clear that

$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \eta_1 = \eta_2 = 0$

To obtain the optimal manufacturing (production) rate and optimal price we differentiate the Lagrange function with respect $u_v(t)$ $P(t)$ to respectively.

$$\frac{\partial L}{\partial P} = 0\ ,\ \frac{\partial L}{\partial u_v} = 0 \qquad (11)$$

$$P\left(t\right)= \frac{-a(t)}{2b}+\frac{\lambda_1(t)}{2}+\frac{\lambda_2(t)}{2} \tag{12}$$

$$u_v\left(t\right) = \hat{u}_v + \frac{\lambda_2}{2} \tag{13}$$

To obtain (the adjoint equations)

$$\dot{\lambda}_1 = \frac{-\partial L}{\partial I_b}\ ,\dot{\lambda}_2 = \frac{-\partial L}{\partial I_v} \tag{14}$$

$$\dot{\lambda}_1 = h_b\left(I_b(t)-\hat{I}_b\right) \tag{15}$$

$$\dot{\lambda}_2 = h_v\left(I_v(t)-\hat{I}_v\right) \tag{16}$$

From equations (2), (3), (14), (15) we can get the following system of linear differential equations.

$$\dot{I}_b = \frac{-a(t)}{2}+\frac{b(t)\lambda_1}{2}+\frac{b(t)\lambda_2}{2} \tag{17}$$

$$\dot{I}_v = \hat{u}_v\ \frac{-a(t)}{2}+\frac{b(t)\lambda_1}{2}+\frac{b(t)\lambda_2}{2}+\frac{\lambda_2(t)}{c} \tag{18}$$

$$\dot{\lambda}_1 = h_b\left(I_b(t)-\hat{I}_b\right) \tag{19}$$

$$\dot{\lambda}_2 = h_v\left(I_v(t)-\hat{I}_v\right) \tag{20}$$

These equations can be written in matrix form as following:

$$\begin{bmatrix}\dot{I}_b\\ I_v\\ \dot{\lambda}_1\\ \dot{\lambda}_2\end{bmatrix} = \begin{bmatrix}0 & 0 & \frac{b}{2} & \frac{b}{2}\\ 0 & 0 & \frac{b}{2} & \frac{b}{2}+\frac{1}{c}\\ h_b & 0 & 0 & 0\\ 0 & h_v & 0 & 0\end{bmatrix}\begin{bmatrix}I_b(t)\\ I_V(t)\\ \lambda(t)_1\\ \lambda_2(t)\end{bmatrix} = \begin{bmatrix}\frac{-a}{2}\\ \hat{u}_v-\frac{a}{2}\\ -\hat{I}_b h_b\\ -\hat{I}_v h_v\end{bmatrix} \tag{21}$$

With the boundary conditions

$$\begin{bmatrix}I_b(0)\\ I_v(0)\\ \lambda_1(T)\\ \lambda_2(T)\end{bmatrix} = \begin{bmatrix}I_b\\ I_v\\ 0\\ 0\end{bmatrix} \tag{22}$$

The equation (2.21) can be expressed in the matrix form as

$$\dot{x} = Ax + b \tag{23}$$

Matrix A has four eigenvalues $_1$,$m_2 m_3$,m_4 and four eigenvectors. The eigenvalues can be obtained from the determinant of characteristic matrix $A - m_i I = 0$.

$$m^4 - \frac{b}{2}h_v\ m^2 - \frac{h_v}{c}\ m^2 + h_b\frac{b}{2}\ m^2 + \frac{b}{2c}\ h_b\ h_v = 0 \tag{24}$$

$$m_i = \pm\sqrt{\frac{-B\pm\sqrt{B^2-4AD}}{2A}} \tag{25}$$

Where

$$A = 1,\ B = \frac{b}{2}\ h_v - \frac{h_v}{c} - h_b\frac{b}{2}\ , D = \frac{b}{2c}\ h_b\ h_v$$

The corresponding characteristic roots m_i of matrix A, there exist non zero vectors v_i such that $(A - m_i I)\ v_i = 0$ where v_i are eigenvectors.

$$\begin{bmatrix} -m_i & 0 & 0.5b & 0.5b \\ 0 & -m_i & 0.5b & 0.5b+\frac{1}{c} \\ h_b & 0 & -m_i & 0 \\ 0 & h_v & 0 & -m_i \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \\ v_{i4} \end{bmatrix} = 0 \tag{26}$$

From above we can get

$$v_{i1} = t$$

$$v_{i2} = \left(\frac{m_i^2}{h_v b} - \frac{h_b}{h_v} \right) t$$

$$v_{i3} = \frac{h_b}{m_i} t$$

$$v_{i4} = \left(\frac{m_i}{b} - \frac{h_b}{m_i} \right) t$$

The general solution consists of homogeneous solution X_h + particular solution X_p .of the system we will use the variation of parameters technique.

$$x_h = \begin{bmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{bmatrix} \begin{bmatrix} c_1 e^{m_1 t} \\ c_2 e^{m_2 t} \\ c_3 e^{m_3 t} \\ c_4 e^{m_4 t} \end{bmatrix} \tag{27}$$

$$x_p = \begin{bmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{bmatrix} \begin{bmatrix} L_1 e^{m_1 t} \\ L_2 e^{m_2 t} \\ L_3 e^{m_3 t} \\ L_4 e^{m_4 t} \end{bmatrix} \tag{28}$$

Let the matrix a is the inverse of matrix v

$$\begin{bmatrix} L_1' e^{m_1 t} \\ L_2' e^{m_2 t} \\ L_3' e^{m_3 t} \\ L_4' e^{m_4 t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{33} & a_{34} \\ a_{41} & a_{24} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \frac{-a}{2} \\ \hat{u}_v - \frac{a}{2} \\ -\hat{I}_b h_b \\ -\hat{I}_v h_v \end{bmatrix} \tag{29}$$

From equation (29) we get L_1', L_2', L_3', L_4' then by using integration we get L_1, L_2, L_3, L_4

The complete optimal solution $X = (x_h) + (x_p)$ is given by ;

$$\begin{bmatrix} I_b(t) \\ I_v(t) \\ \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & v_{31} & v_{41} \\ v_{12} & v_{22} & v_{32} & v_{42} \\ v_{13} & v_{23} & v_{33} & v_{43} \\ v_{14} & v_{24} & v_{34} & v_{44} \end{bmatrix} \left[\begin{bmatrix} c_1 e^{m_1 t} \\ c_2 e^{m_2 t} \\ c_3 e^{m_3 t} \\ c_4 e^{m_4 t} \end{bmatrix} + \begin{bmatrix} L_1 e^{m_1 t} \\ L_2 e^{m_2 t} \\ L_3 e^{m_3 t} \\ L_4 e^{m_4 t} \end{bmatrix} \right] \tag{30}$$

Numerical Example

A numerical example is given for two different states of demand and price rates.

Table 1 presents the values of system parameters and initial states which are used in numerical examples.

Table 1

Parameters	Value
c	$3

h_b	\$1
h_v	\$2
$\hat{u}_v$	15
$\hat{I}_b$	40
$\hat{I}_v$	25
I_{bo}	50
I_{vo}	15
b	0.5
a	30
a	*30+t*
T	5

Where

c, h_b, h_v are the production cost , the buyer holding cost and the vendor holding cost per unit product. Respectively

$\hat{u}_v$, $\hat{I}_b$, $\hat{I}_v$ are the production goal rate , inventory goal level of the buyer and inventory goal level of the vendor .

we summarize the values of the optimal manufacture rate and optimal price and inventory level of buyer and inventory level of vendor at the end of planning horizon time (T) and objective function in table 2.

Table 2 Summary of result at t =T = 5

The results	Demand rate	
	Constant a= 30	Linear a= $30+t$
$I_b(5)$	7	5
$I_v(5)$	30	28
u_v (5)	15	15
P (5)	30	34
Holding cost of buyer	512	566
Holding cost of vendor	74	71
Production cost	113	122
J (optimal)	1064	1325

IV. Conclusions

This study has described the solution of inventory production system using Pontryagin maximum principle. The price control policy and production control policy has maximized the objective function that measure the profit of both buyer and vendor. This model can be extended in many ways .For example transportation cost, order cost, and shortage cost of both buyer and vendor. Also this model can extend to include multiple buyers, multiple vendors and multi-products to represent constraint supply chain problem.

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