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Region Decomposition of Planar Graphs

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Abstract: A graph G is said to be planar if there exist some geometric representation of graph G which can be drawn on a plane without crossover between its edges. In this paper we find the maximal region decomposition of planar graphs by using the reduction rule for dominating set.

Keywords: Planar graph, dominating set, region of graphs

I Introduction

A graph G is said to be planar if there exist some geometric representation of graph G which can be drawn on a plane without crossover between its edges. A set S of vertices of graph G is a dominating set of G if every vertex in V(G)-S is adjacent to some vertex in S. Let G be a planar graph. Embed the graph in plane paper and cut along its edge using a sharp blade, then paper cut into number of pieces , the corners of each pieces are the vertices of G and sides are edges of G and each pieces is called region of graph G.

II The Reduction Rules

We present two reduction rules for Dominating Set. Both reduction rules are based on the same principle. We explore local structures of the graph and try to replace them by simpler structures. For the first reduction rule, the local structure will be the neighborhood of a single vertex . For the second reduction rule, we will deal with the union of the neighborhoods of a pair of vertices.

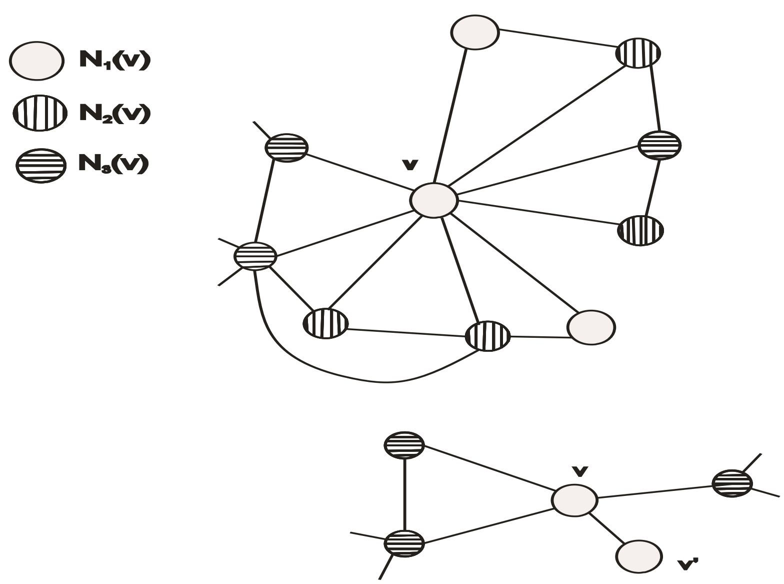

Figure 1: The lower graph shows the partitioning of the neighborhood of a single vertex v.

III The Neighborhood of Single Vertex

Consider a vertex $v \in V$ of the given graph G = (V,E). Here and in the following, for $v \in V$, let $N(v) = \{u : \{u,v\} \in E\}$ be the neighborhood of v. We partition the vertices of N(v) of v into three different sets $N_1(v)$,

N_2(v),and N_3(v) depending on what neighborhood structure these vertices have. More precisely, setting N[v]:=N (v) $\cup \{v\}$, we define N_1(v)= $\{u \in N(v) : N(u) \setminus N[v] \neq \phi\}$,$N_2$(v)= $\{u \in N(v) \setminus N_1(v) : N(u) \cap N_1[v] \neq \phi\}$,

N_3(v) = $N(v) \setminus (N_1(v) \cup N_2(v))$.

An example that illustrates the partitioning of N(v) into the subsets N_1(v), N_2(v), and N_3(v) can be seen in the left-hand diagram of Figure 1.Note that, by definition of the three subsets, the vertices in N_3(v) cannot be dominated by vertices from N_1(v). A good candidate for dominating N_3(v) is given by the choice of v . Observing that this indeed is always an optimal choice lies the base for our first reduction rule.

Rule 1. If $N_3(v) \neq \phi$ for some vertex v, then ---remove N_2(v) and N_3(v) from G and

---add a new vertex v' with edge {v,v'} to G.

We use the vertex v' as a "gadget vertex" that enforces us to take v (or v') into an optimal dominating set in the reduced graph.

Example 1. Figure 1 shows the neighborhood of a vertex v before and after applying Rule 1 to it.

Lemma 1. Let G = (V,E) be a graph and let G' = (V',E') be the resulting graph after having applied Rule 1 to G. Then $\gamma(G') = \gamma(G)$.

Proof. Consider a vertex $v \in V$ such that $N_3(v) \neq \phi$. The vertices in N_3(v) can only be dominated by either v or by vertices in $N_2(v) \cup N_3(v)$. But , clearly, $N(w) \subseteq N(v)$ for every $w \in N_2(v) \cup N_3(v)$. This shows that an optimal way to dominate N_3(v) is given by taking v into the dominating set. This is simulated by the "gadget vertex" v' in G'which enforces us to take v (or v') into an optimal dominating set. It is safe to remove $N_2(v) \cup N_3(v)$ since $N(N_2(v) \cup N_3(v)) \subseteq N(v)$,that is , since the vertices that could be dominated by vertices from $N_2(v) \cup N_3(v)$ are already dominated by v. Hence $\gamma(G') = \gamma(G)$.

IV The Neighborhood of A Pair Of Vertices

Similar to Rule 1, w explore the neighborhood set $N(v,w) := N(v) \cup N(w) \setminus \{v,w\}$ of

two vertices $v, w \in V$. Analogously, we now partition N (v, w) into three disjoint subsets N_1 (v, w), N_2 (v, w), and N_3 (v, w). Setting $N(v,w) := N(v) \cup N(w)$, we define :

$$N_1(v,w) := \{u \in N(v,w) : N(u) \setminus N[v,w] \neq \phi\},$$

$$N_2(v,w) := \{u \in N(v,w) \setminus N_1(v,w) : N(u) \cup N_1(v,w) \neq \phi\},$$

$$N_3(v,w) := N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)).$$

The left-hand diagram of Figure 2 shows an example that illustrates the partitioning of N(v,w) into the subsets N_1 (v, w), N_2 (v, w), and N_3 (v, w).

Our second reduction rule – compared to Rule 1 –is slightly more complicated.

Rule 2: Consider $v, w \in V (v \neq w)$ and suppose that $|N_3(v,w)| > 1$. Suppose that N_3(v,w) cannot be dominated by a single vertex from $N_2(v,w) \cup N_3(v,w)$.

Case 1: If N3 (v, w) can be dominated by a single vertex from {v,w}:

(1.1) If $N_3(v,w) \subseteq N(v)$ as wel $N_3(v,w) \subseteq N(w)$:

---remove N_3 (v, w) and $N_2(v,w) \cap N(v) \cap N(w)$ from G and

---add two new vertices z, z' and edges $\{v, z\}, \{w, z\}, \{v, z'\}, \{w, z'\}$ to G.

(1.2) If $N_3(v,w) \subseteq N(v)$ but not $N_3(v,w) \subseteq N(w)$:

---- remove N_3 (v, w) and $N_2(v,w) \cap N(v)$ from G and

---add a new vertex v' and edge $\{v, v'\}$ to G.

(1.3) If $N_3(v,w) \subseteq N(w)$ but not $N_3(v,w) \subseteq N(v)$:

--- remove N_3 (v, w) and $N_2(v,w) \cap N(w)$ from G and

--- add a new vertex w' and edge $\{w, w'\}$ to G.

Case 2: If N_3 (v, w) cannot dominated by a single vertex from {v, w}:

---remove N_3 (v, w) and N $_2$(v, w) from G and

---add two new vertices v', w' and edges $\{v, v'\}, \{w, w'\}$ to G.

Clearly, Cases (1.2) and (1.3) are symmetric to each other. Again, the newly added vertices v' *and* w' of degree one act as gadgets that enforce us to take v or w into an optimal dominating set. A special situation is given in Case (1.1). Here, the gadget added the graph G simulates that at least one of the vertices v or w has to be taken into an optimal dominating set

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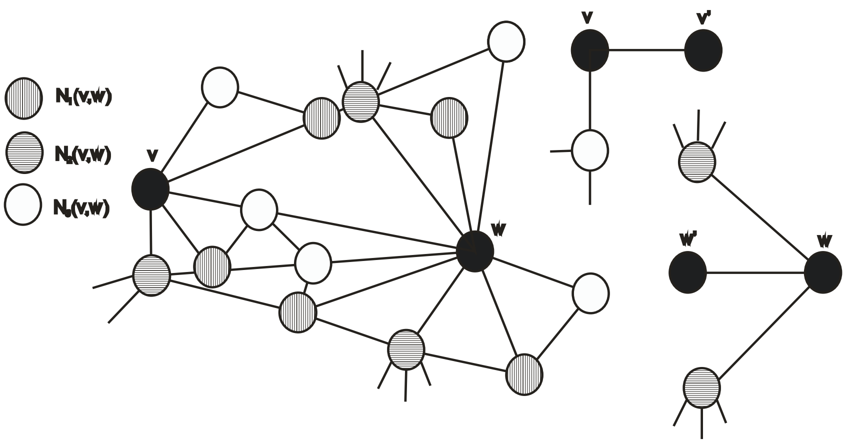

Figure.2. The left hand side shows that the partitioning of a neighborhood N(v,w) of two vertices v and w. The right hand side shows the result of applying Rule 2, Case 2, to this particular (sub) graph.

Lemma 2: Rule 1 can be carried out in O (n) time for planar graphs and in O (n^3) time for general graphs.

Proof : We first discuss the planar case. To carry out Rule 1, for each vertex v of the given planar graph G we have to determine the neighbor sets N_1(v),N_2(v), and N_3(v). By definition of these sets, one easily observes that it is sufficient to consider the sub graph G that is induced by all vertices that are connected to v by a path of length at most two. To do so, we employ a “partial” depth –first search tree of depth two, rooted at v. More precisely, this means that we explore all vertices as distance one from v (i.e., connected to v by an edge in G) and some vertices at distance two from G (to be described in more detail in the following). We perform two phases.

In phase 1, constructing the search tree we determine the vertices from N_1 (v).Each vertex of the first level (i.e., distance one from the root v) of the search tree that has a neighbor at the second level of the search tree belongs to N_1 (v). Observe that it is enough to stop the expansion of a vertex from the first level as soon as its first neighbor in the second level is encountered. Hence , denoting the degree of v by deg(v) , phase 1 takes time O(deg(v)) non-tree edges to be explored. The latter holds true since these non-tree edges all belong to the sub graph of G include by N[v]. Since this graph is clearly planar and $|N[v]| = \deg(v)+1$, the claim follows.

In phase 2, it remains to determine the sets N_2(v) and N_3(v). To get N_2(v), one basically has to go through all vertices from the first level of the above search tree that are not already marked as being in N_1(v) but have at least one neighbor in N_1(v). All this can be done within the planar graph induced by N(v), using the already marked N_1(v)-vertices , in time O(deg(v)). Finally, N_3(v) simply consists of vertices from the first level that are neither marked being in N_1(v) nor marked being in N_2(v). In summary, this shows that for a vertex v the sets N_1 (v), N_2 (v), and N_3 (v) can be constructed in time O(deg(v)).

Once having determine these three sets, the sizes of which all are bonded by deg(v), it is clear that the possible removal of vertices from N_2(v) and N_3(v) and the addition of a vertex and an edge as required by Rule 1 all can be done in time O(deg(v)). Finally, it remains to analyze the overall complexity of this procedure when going through all n vertices of G = (V,E).

But this is easy. The running time can be bounded by $\sum_{v\in V} O(\deg(v))$. Since G is planar, this sum is bounded by O(n), that is, the whole reduction takes linear time. For general graphs, the method described above leads to a worst-case cubic time implementation of Rule 1.Here, one ends up with the sum $\sum_{v\in V} O((\deg(v))^2) = O(n^3)$.

Note that the size of the graph that is induced by the neighborhood N[v] again is relevant for the time needed to determine the sets N_1 (v), N2 (v) and N_3 (v). For general graphs, this neighborhood may contain $O((\deg(v))^2)$ many edges.

Lemma 3. Let G = (V,E) be a graph and let $G' = (V', E')$ be the resulting graph after having applied Rule 2 to G . Then $\gamma(G) = \gamma(G')$.

Proof. Similar to the proof of Lemma 1, we observe that vertices from $N_3(v,w)$ can only be dominated by vertices from $M := \{v,w\} \cup N_2(v,w) \cup N_3(v,w)$. All cases in Rule 2 are based on the fact that $N_3(v,w)$ needs to be dominated. All cases only apply if there is not a single vertex in $N_2(v,w) \cup N_3(v,w)$ which dominates $N_3(v,w)$.

We first of all discuss the correctness of Case (1.2) (and similarly obtain the correctness of the symmetric Case (1.3)): If v dominates $N_3(v,w)$ (and w does not) then it is optimal to take v into the dominating set – and at the same time still leave the option of taking vertex w- than to take any combination of two vertices x,y from the set M\{v}.It may be that we still have to take w to get a minimum dominating set , but any case v and w dominate at least as many vertices as x and y . The “gadget edge” {v,v’} simulates the effect of taking v. It is safe to remove $R := (N_2(v,w) \cap N(v)) \cup N_3(v,w)$ since , by taking v into the dominating set, all vertices in R are already dominated and since , as discussed above ,it is always at least as good to take v into a minimum dominating set than to take any other of the vertices from M.

In the situation of case (1.1) , we can dominate $N_3(v,w)$ by both either v or w. Since we cannot decide, at this point, which of these vertices should be chosen to be in the dominating set ,we use the gadget with vertices z and z’ , which simulates a choice choice between v or w , as can be seen easily. In any case, however, it is at least as good to take one of the vertices v and w (may be both) than to take any other two vertices from M. The argument for this is similar to the one for Case (1.2). The removal of $N_3(v,w) \cup (N_2(v,w) \cap N(v) \cap N(w))$ is safe by a similar argument as the one that justified the removal of R in Case (1.2).

Finally, in Case 2, we clearly need at least two vertices to dominate $N_3(v,w)$. Since $N(v,w) \supseteq N(x,y)$ for all pairs of $x, y \in M$ it is optimal to take v and w into dominating set, simulated by the gadgets {v,v’} and {w,w’}. As in the previous cases the removal of $N_3(v,w) \cup N_2(v,w)$ is safe since these vertices are already dominated and since these vertices need not be used for an optimal dominating set.

It is easy to see that applying the reduction rules to planar graphs always results in a planar graph again. This is due to the fact that the removal of vertices and edges does not affect planarity and the gadget vertices (and edges) that are introduced by

Rules 1 and 2 clearly can be drawn without causing edge crossings. Here, only Case (1.1) of rule 2 needs a little care: Since $N_3(v,w) \subseteq N(v)$ as well as $N_3(v,w) \subseteq N(w)$, the removal of $N_3(v,w)$ provides "space" for the (clearly planar) gadget drawn between v and w without any edge crossings.

Lemma 4. Rule 2 can be carried out in time $O(n^2)$ for planar graphs and in time $O(n^4)$ for general graphs.

Proof. To prove the time bounds for Rule 2, basically the same ideas as for Rule 1 apply (cf. proof of lemma 2). In stead of a depth –two search trees, one now has to argue on a search tree where the levels indicate the minimum of distances to vertex to vertex v or w. Hence, we associate the vertices v and w to the root of this search tree. The first level consists of all vertices that lie in N(v,w) (i.e., at distance one from either of the vertices v or w). Determining the subset $N_1(v, w)$ means to check whether some vertex on the first level has a neighbor on the second level .We do the same kind of construction as in Lemma 2, the running time is determined by the size of sub graph induced by the vertices that correspond to the root and the first level of this search tree, that is, by G[N[v , w]] in this case. For planar graphs, we have $|G[N[v,w]]|$ = O (deg (v) + deg (w)). Hence, we get $\sum_{v,w\in V} O(\deg(v) +$ deg (w)) as an upper bound on the over all running time in the case of planar graphs. Making use of the fact that $\sum_{v\in V}\deg(v)$=O(n) for planar graphs, this upper bounded by O $\left(\sum_{v,w\in v}\deg(v)+\sum_{v,w\in v}\deg(w)\right)$=O($n^2$).In case of general graphs, we have $|G[N[v,w]]|$=O((deg(v) + deg(w))^2),which trivially yields the upper bound $\sum_{v,w\in V} O((\deg(v)$ + deg(w))^2) = O(n^4) for the overall running time.

V Finding A Maximal Region Decomposition

Suppose that we have a reduced planar graph G with minimum dominating set D. we know that, in particular, neither Rule 1 applies to a vertex $v\in D$ nor Rule 2 applies to a pair of vertices v, $w\in D$. we want to get our hands on the number of vertices which lie in neighborhood N(v ,w) for v , $w\in D$. A first idea to prove that $|V|$ = O($|D|$) would be to find (ℓ = O($|D|$) many neighborhoods N(v_1, w_1),………, $N(v_\ell, w_\ell)$ with $v_i, w_i \in D$ such that all vertices in V lie in at least one such neighborhood; and then use the fact that G is reduced in order to prove that each N(v_i , w_i) has size O(1).even if the graph G is reduced, however ,the neighborhoods $N(v,w)$ of two vertices $v,w\in D$ may contain many vertices: the size of $N(v,w)$ in a reduced graph basically depends on how big $N_1(v,w)$ is.

In order to circumvent these difficulties, we define the concept of region $R(v,w)$ for which we can guarantee that in a reduced graph it consists of only a constant number of vertices.

Definition 1: Let $G=(V,E)$ be a planar graph. A region $R(v,w)$ between two vertices v,w is a closed subset of the plane with the following properties :

(1) the boundary of $R(v,w)$ is formed by two simple path P_1 and P_2 in V that connect v and w ,and the length of each path is most three[7],and

(2) all vertices that are strictly inside[8] the region $R(v,w)$ are from $N(v,w)$.

For a region $R=R(v,w)$, let $V(R)$ denote the vertices belonging to R , that is, $V(R) := \{u\in V : u$ sits inside or on the boundary of $R\}$.

In the following , the boundary of a region R will be denoted by ∂R.

Definition 2: Let $G=(V,E)$ be a plane graph and $D\subseteq V$. A $D-region\,decomposition$ of G is a set R of regions between pairs of vertices in D such that

(1) for $R(v,w)\in R$ no vertex from D (except for v,w) lies in $V(R(v,w))$ and

(2) for two regions $R_1, R_2 \in R$,it holds $(R_1\cap R_2)\subseteq(\partial R_1 \cup \partial R_2)$.

For a D-region decomposition R, we define $V(R) := \bigcup_{R \in R} V(R)$. A D-region decomposition R is called maximal if there is no region $R \notin R$ such that
$R' := R \cup \{R\}$ is a D-region decomposition where $V(R)$ is a strict subset of $V(R')$.
For an example of a (maximal) D-region decomposition, we refer to the left-hand side diagram of Figure 3.

We will show that, for a given graph G with dominating set D, we can always find a maximal D-region decomposition induces a graphs in very natural way.

Definition 3: The induced graph $G_R = (V_R, E_R)$ of a D-region decomposition R of G is the graph with possible multiple edges that is define by $V_R := D$ and
$E_R := \{\{v,w\} :$ there is a region $R(v,w) \in R$ between $v, w \in D\}$.

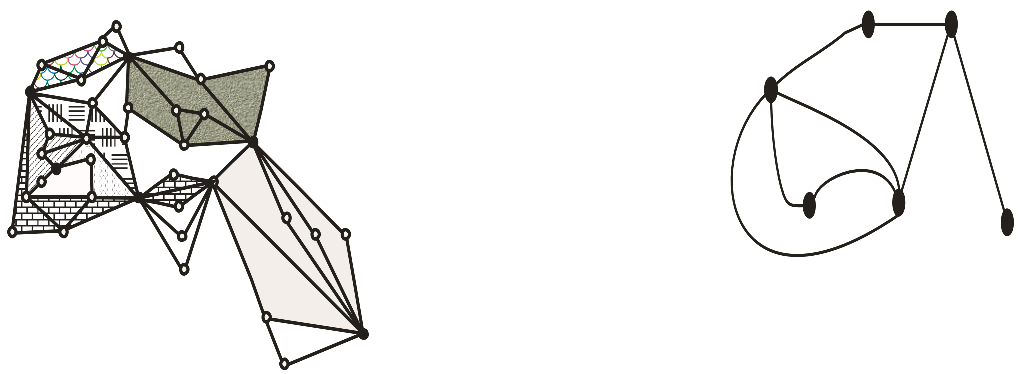

Figure 3. The left-hand side diagram shows an example of a possible D-region decomposition R of some graph G, where D is the subset of vertices in G that are drawn in black.

The various regions are high lightened by different patterns. The remaining white areas are not considered as regions. The given d-region decomposition is maximal. The right- hand side shows the induced graph G_R (Definition 4.3).
Note that, by Definition 4.3, the induced graph G_R of D-region decomposition is planar. For an example of an induced graph $G_{R,}$ see Figure 3.

Definition 4: A planar graph G = (V,E) with multiple edges is thin if there exists a planar embedding such that no two multi edge are hemitropic; This means that if there are two edges e_1,e_2 between a pair of distinct vertices $v, w \in V$, then there must be two further vertices u_1,$u_2 \in V$ that sit inside the two disjoint areas of the plane that are enclosed by e_1,e_2.
The induced graph G_R in figure 3 is thin.

Lemma 5. For a thin planar graph G = (V,E) , we have $|E| \le 3|V| - 6$.

Proof. The claim is true for planar graphs without multiple edges. We prove the claim by an induction on the number ℓ_G of multiple edges in G. More precisely, for a graph G = (V,E) with multiple edges (i.e., E is a multi set), we let $\ell_G := \frac{1}{2}\left(\sum_{v,w \in V}\left(\left(\sum_{\{v,w\} \in E} 1\right) - 1\right)\right)$, For ℓ_G =0,the claim is true, since a planar graph (without multiple edges) has at most $3|v| - 6$ edges. Now, suppose the claim is true for all graphs which have at most ℓ_G multiple edges. Consider a planar graph G = (V, E) with $\ell_G + 1$ multiple edges.

Choose a pair of vertices $v, w \in V$ that is connected by at least two edges e_1, $e_2 \in E$. Since G is thin, we may consider a planar embedding, in which e_1 and e_2 are not homotopic. Let G_1= (V_1,E_1) be the sub graph of G that consists of the vertices v, w the edge e_1 and all vertices and edges that sit strictly inside the area A of the plane that is enclosed by $e_1$and e_2 . similarly, let G_2 = (V_2,E_2) be the sub graph of G that consists of the vertices v,w, the edge e_2 and all vertices and edges that sit strictly outside the area a. Hence, we have $|E| = |E_1| + |E_2|$ and $|V| = |V_1| + |V_2| - 2$ .Since, by construction, $\ell_{G_1}, \ell_{G_2} < \ell_G$, the induction hypothesis yields $|E| = |E_1| + |E_2|$

$$\leq (3|V_1| - 6) + (3|V_2| - 6)$$
$$= 3|V| - 6$$

Using the notion of thin graphs, we can formulate the main result of this subsection.

Proposition 1: For a reduced plane graph G with dominating set d, there exists a maximal D-region decomposition R such that G_R is thin.

Proof. We give a constructive proof on how to find a maximal D-region decomposition R of a plane graph G such that the induced graph G_R is thin. It is obvious that the D-region decomposition, since by construction we made sure that regions are between vertices in D , that regions do not contain vertices from D , and that regions do not intersect. Moreover, the D-region decomposition obtained by the algorithm is maximal: If a vertex u does not belong to region, that is , if $u \notin V_{used}$, then the algorithm eventually checks, whether there is a region S_u such that $R \cup \{S_u\}$ is a D-region decomposition.

It remains to show that the induced graph G_R of the d-region decomposition R found by the algorithm is thin. We embed G_R in the plane in such a way that an edge belonging to a region S $\in$ R is drawn inside the area covered by R. to see that the graph is thin , we have to show that , for every multiple edge e_1 , e_2 (belonging to two regions S_1 ,$S_2 \in R$ that were chosen at some point of the algorithm) between two vertices v,w $\in D$, there exist two vertices u_1 , u_2 $\in D$ that lie inside the areas enclosed by e_1, e_2. Let A be such an area. Suppose that there is no vertex u $\in D$ in A. We distinguish two cases. Either there is also no vertex from V\D in A or there are other vertices V' from V\D inside A. In the first case, by joining the regions $R_1$and R_2 we obtain a bigger region which fulfills all the four conditions checked by the algorithm in Figure 4.a contradiction to maximality of $R_1$and R_2. In the second case, since D is assumed to be a dominating set, the vertices in V' need to be dominated by D. Since v,w are the only vertices from D which are part of A, R_1 or R_2, the vertices in V'need to be dominated by v,w, hence they belong to N(v, w).

VI Concluding Remarks

In this work, two lines of research meet. On the one hand ,there is DOMINATING Set ,One of the NP-complete core problem of combinatorial optimization and graph theory .On the other hand ,the second line of research is that of algorithm engineering and ,in particular , the power of data reduction by efficient preprocessing presenting two simple and easy to implement reduction rules for DOMINATING Set, we proved that for planar graph a linear size problem kernel can be efficiently constructed. It might be interesting see whether similar reduction rules with a provable guarantee on the of the reduced instances can also be found variation of DOMINATING Set problem ,such as TOTAL DOMINATING Set, or PERFECT DOMINATING Set.

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