g-Pre Regular And g-Pre Normal Topological Spaces

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Abstract - In general topology, the notion of pre-open set, introduced by A.S. Mashhour et al. [1982], has a significant role and the most important generalizations of regularity & normality appear as the notions of pre-regularity along with strong regularity [1983] and pre-normality as well as strong normality [1984] respectively.

In 1970, N. Levine projected the concept of so called g-closed sets in topological spaces in an independent way and studied its basic properties. Since then many modifications of g-closed sets were defined and investigated by a large number of topologists. In 1996, Maki et al. introduced the concepts of gp-closed sets. The purpose of this paper is to study the classes of regular spaces & normal spaces, namely gp-regular spaces & gp-normal spaces which are a generalization of the classes of p-regular & p-normal spaces respectively. The paper also contains the behaviour of pre * —T\textsubscript{1/2} spaces whenever it is strongly regular or strongly normal. It also highlights the pre-topological property of a gp-normal pre-\textit{R}_0 spaces.

Also, through this paper, a tribute is being paid to the renowned mathematician Professor M.E. Abd. El – Monsef who left for his heavenly abode on 13\textsuperscript{th} August, 2014.

Keywords - g-pre –regular spaces, g-pre –normal spaces, g*\textit{K} – regular spaces, g*\textit{K} - normal spaces.

I. INTRODUCTION & PRELIMINARIES

Various new topological concepts & their basic properties have been defined & investigated using the notion of pre-open sets & pre-open, pre-continuous mappings (i.e. pre homeomorphism) as introduced by A.S. Mashhour et al. [1]. In 1998, T. Noiri et al. [2] studied generalized pre closed functions using generalized preclosed sets.

A subset A of a space (X, T) is known as a generalized pre-closed iff every open superset of A contains its pre-closure[2].

In the present paper, spaces(X,T) and (Y,\sigma) always mean topological spaces which are not assumed to satisfy any separation axioms are assumed unless explicitly mentioned.

Also, f: (X,T)\rightarrow (Y,\sigma) denotes a single valued function f of a space (X,T) into another space (Y,\sigma). And for a subset A of a space (X,T), X/A = A\textsuperscript{c}, cl(A) & int(A) denote the complement, the closure & the interior of A in (X,T) respectively. The concepts of g-pre –regular & g-pre –normal spaces are, here, studied using generalized pre-closed sets.

Definition (1.1): A subset A of a topological space (X,T) is called

1. regular open or open domain[1] if A = int(cl(A)).
2. an \alpha-open[3] set if A \subseteq int(cl(int(A)))
3. pre-open [4] or nearly open[1] set if A \subseteq int(cl(A)).

The compliments of the above mentioned open sets are their respective closed sets. The smallest \textit{K}-closed set containing A is called \textit{K}cl(A) where \textit{K} = regular, \alpha, p, s, \beta & \pi. The largest \textit{K}-open set contained in A is called \textit{K}int(A) where \textit{K} = regular, \alpha, p, s, \beta & \pi.

The family of all \textit{K}-open (resp. \textit{K} -closed) sets of a space (X,T) is denoted by \textit{K}O(X)(resp. \textit{K}C(X)); here and above \textit{K} = regular, \alpha, p, s, \beta & \pi.

Definition(1.2)[1,5]: (a) The p-interior of a subset A, denoted as p-int(A), is defined as the union of all p-open sets contained in A.

(b) The pre - closure of subset A, denoted as p - cl(A), is defined as the intersection of all p - closed sets containing A. Naturally, pnt(A) is pre open where as pcl(A) is pre closed where A is subset of X.

Also, pnt(A) = A \cap int(cl(A)) & pcl(A) = A \cap cl(int(A)).

Definition (1.3)[2]: A subset A of a space (X,T) is said to be generalized preclosed (briefly gp-closed) iff pcl(A) \subseteq U whenever A \subseteq U and U is open in X.

Definition (1.4)[2]: For a subset A of a space (X,T)

 pcl(A) = \cap \{ A \subseteq F, F is gp-closed in X \} & gpnt(A) = \cup \{ V \subseteq A, V is gp-open in X \}

also, gpcl(A) \subseteq pcl(A) as every pre closed set is gp- closed.

Definition (1.5)[6]: A space (X,T) is called pre * —T\textsubscript{1/2} if every gp-closed set in X is preclosed.
Definition (1.6): A function \( f : (X,T) \rightarrow (Y,\sigma) \) is called pre continuous [1] (resp. pre irresolute[7]) if the inverse image of each open(resp. pre open) set of \( Y \) is pre open in \( X \).

Definition (1.7)[8]: A bijective function \( f : (X,T) \rightarrow (Y,\sigma) \) is called pre-homeomorphism if \( f \) is M-pre open and pre-irresolute.

Definition (1.8) [5]: A space \((X,T)\) is called strongly regular if for each pre closed set \( A \) & each point \( x \notin A \), there exist pre-open sets \( U \) & \( V \) such that \( X \subseteq U \cup A \subseteq V \).

Definition (1.9)[9]: A space \((X,T)\) is called strongly normal if for each pair of disjoint pre-closed sets \( A \) & \( B \), there exist pre-open sets \( U \) & \( V \) such that \( A \subseteq U \cup B \subseteq V \).

Any other notation and symbol, not defined in this paper, may be found in the appropriate reference.

II. g-PRE REGULAR SPACES

This section introduces g-pre regular spaces in topological spaces.

Definition (2.1): A topological space \((X,T)\) is said to be g-pre-regular (in short gp-regular) space iff every gp-closed set \( F \) and every point \( x \notin F \), there exist disjoint pre-open sets \( U \) & \( V \) such that \( F \subseteq U \cup x \in V \).

Lemma (2.2): A strongly regular pre \(-T_{1/2}\) space is gp-regular.

Proof: Let \((X,T)\) be any g-pre-regular space as well as \( -T_{1/2} \) space. Since, \((X,T)\) is a pre \(-T_{1/2}\) space, hence every gp-closed set in \( X \) is pre closed i.e. the class of gp-closed sets & pre-closed sets coincide. Now, \((X,T)\) is strongly regular space which provides that for each pre closed set \( A \) & each point \( x \notin A \) , there exist disjoint pre-open sets \( U \) & \( V \) such that \( x \in U \cup A \subseteq V \). Combining these facts, it is concluded that for each gp-closed set \( A \) and each point there exist disjoint pre-open sets \( U \) & \( V \) such that \( A \subseteq U \cup x \in V \), which turns \((X,T)\) to be a gp-regular.

Characterization criteria:

Theorem (2.3): A topological space \((X,T)\) is gp-regular iff every gp-closed set \( F \) and every point \( x \notin F \), there exists pre-open sets \( U \) & \( V \) such that \( x \in U \subseteq F \subseteq V \).

Proof: Suppose that \( F \) is a gp-closed set of a space \((X,T)\) and \( x \notin F \). Since, \((X,T)\) is a gp-regular space hence, there exist disjoint pre-open sets \( U \) & \( V \) such that \( F \subseteq U \cup V \) & \( x \in U \cup U \cap V = \phi \). Obviously, \( U \cup V = \phi \) \( \Rightarrow U \cap (pcl(V)) = \phi \) & \( pcl(U) \cap pcl(V) = \phi \). Converse is not natural, so omitted.

Theorem (2.4): For a space \((X,T)\) the following are equivalent:

(i) \((X,T)\) is gp-regular.

(ii) for every \( x \in X \) and for every gp-open set \( W \) containing \( x \), there exists a pre-open set \( V \) such that \( pcl(V) \subseteq W \).

(iii) for every gp-closed set \( F \) and every point \( x \notin F \), there exists pre-open set \( V \) such that \( pcl(V) \cap F = \phi \).

Proof: (i) \( \Rightarrow \) (ii):

Let \((X,T)\) be a gp-regular space. Let \( W \) be a gp-open set containing a point \( x \in X \). Since, \( W^C \) is gp-closed set and \( x \notin W^C \), hence by the hypothesis, there exist pre-open sets \( U \) & \( V \) such that \( W^C \subseteq U \cup V \) & \( x \in U \cup U \cap V = \phi \).

Now, \( U \cup V = \phi \) \( \Rightarrow U \subseteq U^C \)

\( \Rightarrow pcl(V) \subseteq pcl(U^C) = U^C \)

Again, \( W^C \subseteq V \) \( \Rightarrow U^C \subseteq W \). Combining these two relations, we get \( pcl(V) \subseteq W \).

(ii) \( \Rightarrow \) (i):

Let \( F \) be any gp-closed set and \( x \notin F \). Then \( x \in F^C \) and \( F^C \) is a gp-open set. By hypothesis, there exists a pre-open set \( V \) of \( x \) such that \( pcl(V) \subseteq F^C \), obviously, \( F \subseteq (pcl(V))^C \) and \( (pcl(V))^C \) is a pre-open set containing \( F \) and \( V \cap pcl(V)^C = \phi \).

Therefore, \((X,T)\) is gp-regular.

(iii) \( \Rightarrow \) (iii):

Let \( x \in X \) and \( F \) be a gp-closed set such that \( x \notin F \). Then \( x \in F^C \) and \( F^C \) is a gp-open set. By hypothesis, there exists a pre-open set \( V \) of \( x \) such that \( pcl(V) \subseteq F^C \). Clearly, \( pcl(V) \cap F = \phi \).

(iii) \( \Rightarrow \) (ii):

Let \( x \in X \) and \( W \) be a gp-open set containing \( x \). Since, \( W^C \) is gp-closed set and \( x \notin W^C \), hence, by hypothesis, there exists a pre-open set \( U \) containing \( x \) such that \( pcl(U) \cap W^C = \phi \).

Therefore, \( pcl(V) \subseteq W \).

Hereditary property: The following lemmas are helpful in analyzing the hereditary property of gp-regular spaces:

Lemma (2.5): If \( X_0 \in oO(X,T) \) and \( A \in PO(X,T) \), then \( X_0 \cap A \in PO(X_0, T_{X_0}) \).

Lemma (2.6)[10]: Suppose \( B \subseteq X \) \& \((X,T)\) is a space. If \( A \) is open & gp-closed in \((X,T)\) and \( B \) is a gp-closed in \((A, T_x)\), then \( B \) is also gp-closed in \((X,T)\).

Theorem (2.7): If \((X,T)\) is a gp-regular space & \( Y \) is an open and gp-closed subset of \((X,T)\), then the subspace \((Y,T_Y)\) is a gp-regular space.

Proof: Let \( F \) be any gp-closed subset of \((Y,T_Y)\) and \( y \notin F \) so that \( y \in F^C \). Since, \( Y \) is an open & gp-closed set in \((X,T)\), hence, in view of Lemma (2.6), \( F \) is gp-closed in \((X,T)\).
Since, \((X,T)\) is gp-regular, then there exist disjoint pre-open sets \(U \& V\) of \((X,T)\) such that \(y \in U \& F \subseteq V\).

As \(Y\) is also open so \(Y\) is \(\alpha\)-open and consequently by lemma (2.5), we get \(U \cap Y \& V \cap Y\) as disjoint pre-open sets of the subspace \((Y,T_y)\) such that \(y \in U \cap Y \& F \subseteq U \cap Y\). Hence, \((Y,T_y)\) is a gp-regular space.

Hence, the theorem.

**Preservation theorem:** The gp-regularity of a space is preserved under a bijective, gp irresolute and M-pre-open mapping as established in the following theorem.

**Theorem (2.8):** If \(f : (X,T)\to(Y,\sigma)\) be a bijective, gp irresolute and M—pre-open mapping from a gp-regular \((X,T)\), then \((Y,\sigma)\) is also gp-regular.

**Proof:** Let \(f : (X,T)\to(Y,\sigma)\) be a bijective, gp irresolute and M—pre-open mapping from a gp-regular \((X,T)\) to another space \((Y,\sigma)\).

Let \(y \in Y\) and \(F\) be any gp-closed subset of \((Y,\sigma)\) with \(y \notin F\). Recall that a map \(f : (X,T)\to(Y,\sigma)\) is known to be gp-irresolute if \(f^{-1}(S)\) is gp-closed in \(X\) for every gp-closed set \(S\) in \(Y[2]\).

Hence, \(f^{-1}(F)\) is a gp-closed in \((Y,\sigma)\). Since, \(f\) is bijective, let \(f(x) = y\), then \(x \notin f^{-1}(y)\). By hypothesis, there exist pre-open sets \(U \& V\) such that \(x \in U\) and \(f^{-1}(F) \subseteq Y\) with \(U \cap V = \emptyset\). Since, \(f\) is M-pre-open and bijective, we have \(y \in f(U)\) and \(F \subseteq f(V)\) and \(f(U) \cap f(V) = f(U \cap V) = \emptyset\). Hence, \((Y,\sigma)\) is a gp-regular space. Hence, the theorem.

### III. **g-Pre normal spaces**

The weak form of normality called gp-normality in topological spaces is being introduced and studied in this section.

**Definition (3.1):** A topological space \((X,T)\) is said to be gp-normal (in short gp-normal) space iff for any pair of disjoint gp-closed sets \(A \& B\), there exist disjoint pre-open sets \(U \& V\) such that \(A \subseteq U \& B \subseteq V\).

Transformation of gp-normal space into a gp-regular space occurs only when it is a pre-\(R_0\) space as described through the following theorem (3.2).

**Theorem (3.2):** Every gp-normal, pre-\(R_0\) space is gp-regular.

**Proof:** Let \((X,T)\) be gp-normal as well as pre-\(R_0\) space. Let \(A\) be any gp-closed set in \((X,T)\) and \(x \in X\) with \(x \notin A\).

Now, as \((X,T)\) is pre-\(R\), so \(x\) is gp-closed in \((X,T)\). Since, \((X,T)\) is gp-normal, hence, there exist disjoint pre-open sets \(U \& V\) in the manner that \(x \in \{U\} \subseteq U \& A \subseteq V\). Consequently, \((X,T)\) is gp-regular.

**Characterization criteria:** The following theorems are enunciated to characterize a gp-normal space.

**Theorem (3.3):** A topological space \((X,T)\) is gp-normal iff every pair of disjoint gp-closed sets \(A \& B\) there exist a pair of pre-open sets \(U \& V\) such that \(A \subseteq U \& B \subseteq V\). and pcl\((U) \cap pcl\((V) = \emptyset\).

**Proof:** Let \((X,T)\) be a gp-normal space, and \(A \& B\) are any two disjoint gp-closed sets. Then the gp-normality provides that there exist a pair of disjoint pre-open sets \(U \& V\) in the manner that \(A \subseteq U \& B \subseteq V\).

Now, \(U \subseteq pcl\((U) \& V \subseteq pcl\((V) \& U \subseteq V^c \& V \subseteq C^c\) provide that \(pcl\((U) \cap V = \emptyset\) & pcl\((V) \cap U = \emptyset\) \Rightarrow pcl\((U) \cap pcl\((V) = \emptyset\).

Conversely, Let \(A \& B\) are any pair of disjoint gp-closed sets. By the hypothesis, there exist pre-open sets \(U \& V\) in the manner that \(A \subseteq U \& B \subseteq V\) and \(pcl\((U) \cap pcl\((V) = \emptyset\).

Of course, \(pcl\((U) \cap pcl\((V) = \emptyset\) \Rightarrow U \cap V \subseteq pcl\((U) \cap pcl\((V) = \emptyset\) \Rightarrow U \cap V = \emptyset\).

Hence, \((X,T)\) is gp-normal.

**Theorem (3.4):** For a space \((X,T)\) the following are equivalent:

(i) \((X,T)\) is gp-normal.

(ii) for every gp-closed set \(F\) and every open set \(G\) containing \(F\), there exists a pre-open set \(V\) such that \(F \subseteq U \subseteq pcl\((U) \subseteq G\).

**Proof:** (i) \(\Rightarrow\) (ii):

Let \((X,T)\) be gp-normal space. Let \(F\) be a gp-closed set and \(G\) be a gp-open set such that \(F \subseteq G\) then, \(F \cap G^c = \emptyset\).

Now, \(F \& G^c\) are disjoint gp-closed sets in a gp-normal space \((X,T)\). So there exist pre-open sets \(U \& V\) in the manner that \(F \subseteq U \& G^c \subseteq V\) where \(U \cap V = \emptyset\).

Obviously, \(V^c \subseteq G \& U \cap V = \emptyset \Rightarrow U \subseteq V^c\). As \(V^c\) is pre-closed so \(pcl\((U) \subseteq V^c\).

Therefore, \(F \subseteq U \subseteq pcl\((U) \subseteq G\) holds good.

(ii) \(\Rightarrow\) (i):

Given that for every gp-closed set \(F\) and every open set \(G\) containing \(F\), there exists a pre-open set \(V\) such that \(F \subseteq U \subseteq pcl\((U) \subseteq G\) where \(F \& G^c\) are subsets of a space \((X,T)\).

Now, \(pcl\((U) \subseteq G \Rightarrow G^c \subseteq pcl\((V) \subseteq G^c \Leftrightarrow \emptyset\).\) also, \(U \cap \{pcl\((V) = \emptyset\). Thus, \(F \& G^c\) are disjoint gp-closed sets in \((X,T)\) and \(U \& pcl\((V) = \emptyset\) are disjoint pair of pre-open sets.

Obviously, \(F \subseteq U \& G^c \subseteq pcl\((V) \subseteq V^c\) i.e. for a pair of disjoint gp-closed set \(F = A \& G^c = B\), there exist a pair of disjoint pre-open sets \(U \& V = pcl\((V) \subseteq V^c\) in the manner that \(A \subseteq U\) and \(B \subseteq V\) where \(U \cap V = \emptyset\). hence, \((X,T)\) is gp-normal space.

**Hereditary criteria:** gp-normality is hereditary property with respect to an open and gp-closed subspace.

**Theorem (3.5):** If \((X,T)\) is a gp-normal space and \(Y\) is an open & gp-closed subset of \((X,T)\), then \((Y,T_y)\) is a gp-normal subspace.

**Proof:** Let \(A \& B\) be any two disjoint gp-closed sets of \((Y,T_y)\).

Since, \(Y\) is an open & gp-closed set in \((X,T)\), hence, in view of Lemma (2.6), \(A \& B\) are gp-closed in \((X,T)\).

Since, \((X,T)\) is gp-normal, then there exist disjoint pre-open sets \(U \& V\) of \((X,T)\) such that
As Y is also open so Y is α-open and consequently by lemma (2.5), we get \( U \cap Y \) and \( V \cap Y \) as disjoint pre-open sets of the subspace \((Y, T_Y)\) such that \( A \in U \cap Y \) and \( B \subseteq U \cap Y \). Hence, \((Y, T_Y)\) is a gp-normal space.

**Preservation criteria:** The gp-normality of a space is preserved under a bijective, gp-irresolute and \( M^- \)-pre-open mapping as expressed in following theorem.

**Theorem (3.6):** If \( f : (X,T) \rightarrow (Y,\sigma)\) be a bijective, gp-irresolute and \( M^- \)-pre-open mapping from a gp-normal \((X,T)\), then \((Y,\sigma)\) is also gp-normal.

**Proof:** Let \( f : (X,T) \rightarrow (Y,\sigma)\) be a bijective, gp-irresolute and \( M^- \)-pre-open mapping from a gp-normal \((X,T)\) to another space \((Y,\sigma)\). Let \( A \) and \( B \) be a pair of disjoint gp-closed sets of \((Y,\sigma)\). Since, the map \( f \) is gp-irresolute, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint gp-closed sets of \((X,T)\). As \((X,T)\) is gp-normal, there exist pre-open sets \( U \) and \( V \) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Since, \( f \) is \( M^- \)-pre-open and bijective, we have \( f(U) \) and \( f(V) \) are pre-open sets in \((Y,\sigma)\) such that \( A \subseteq f(U) \) and \( B \subseteq f(V) \). Hence, \((Y,\sigma)\) is a gp-normal space. Hence, the theorem.

IV. **CONCLUSION**

The generalized pre closed sets are used to introduce the concepts gp-regular & gp-normal space. Also, the characterization, the preservation & hereditary nature of gp-regular as well as gp-normal spaces have been framed and analyzed.

Transformation of a strongly regular space into a gp-regular space under the criteria of being \( M^{-} \) has been discussed.

Transformation of a gp-normal space into a gp-regular space under the criteria of being \( \text{pre-R}_0 \) has also been analyzed.

Of course, the entire content will be a successful tool for the researchers for finding the path to obtain the results in the context of \( g \mathcal{K} \)- regular / normal spaces where \( \mathcal{K} = p, b, \beta \).

**REFERENCES**


