Volume 5, Issue 10, October-2015 ISSN: 2277 128X
International Journal of Advanced Research in Computer Science and Software Engineering
Research Paper
Available online at: www.ijarcsse.com

Unidominating Functions of a Cycle

V. Anantha Lakshmi*
Department of Mathematics,
P. R. Govt. College (A), Kakinada,
Andhra Pradesh, India

B. Maheswari
Department of Applied Mathematics,
S. P. Women's University, Tirupati,
Andhra Pradesh, India

Abstract— ***Graph Theory has innumerous applications in different areas such as physical sciences, biological sciences, social sciences, linguistics and other branches of Mathematics. The theory of domination in graphs is a rapidly growing area of research in Graph Theory at present. Recently dominating functions in domination theory have received much attention. In this paper we present some unidominating functions of a cycle and determine the unidomination number. Further we determine the number of unidominating functions of minimum weight for cycles.***

Keywords— ***Cycle, Unidominating function, Unidomination number.***

I. INTRODUCTION

Graph Theory plays an important role in several areas of computer science such as switching theory, logical design, artificial intelligence, formal languages, computer graphics etc. Domination and its properties have been extensively studied by T.W.Haynes and others in [1], [2]. Hedetniemi [3] introduced the concept of dominating functions. We have introduced the new concept of unidominating function and studied unidominating functions for paths [4].

In this paper the unidominating functions of a cycle are studied and obtained the unidomination number. Also the number of unidominating functions of minimum weight for cycles is determined. Further the results obtained are illustrated.

II. UNIDOMINATING FUNCTIONS AND UNIDOMINATION NUMBER

In this section the concepts of unidominating function, unidomination number are defined as follows:

Definition 1: Let $G(V,E)$ be a graph. A function $f: V \to \{0,1\}$ is said to be a Unidominating function if

$$\sum_{u \in N[v]} f(u) \geq 1 \quad \forall\, v \in V \ and\ f(v) = 1,$$

$$\sum_{u \in N[v]} f(u) = 1 \quad \forall\, v \in V \ \ and \ \ f(v) = 0.$$

Definition 2: The unidomination number of a graph $G(V,E)$ is the $min\ \{f(V)/f \text{ is a unidominating function}\}$. It is denoted by $\gamma_u(G)$.

III. UNIDOMINATION NUMBER OF A CYCLE

In this section we find the unidomination number of a cycle and the number of unidominating functions of minimum weight for a cycle.

Theorem 3.1: The unidomination number of a cycle C_n is

$$\gamma_u(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if\ \ n \equiv 0,1(mod\ 3), \\ \left\lceil \frac{n}{3} \right\rceil + 1 & if\ \ n \equiv 2(mod\ 3). \end{cases}$$

Proof: Let C_n be a cycle with vertex set $V = \{v_1, v_2, \dots\dots.., v_n\}$.

To find the unidomination number of a cycle C_n the following three cases arise.

Case 1: Let $n \equiv 0(mod\ 3)$.

Define a function $f: V \to \{0,1\} by$

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2(mod\ 3), \\ 0 & otherwise. \end{cases}$$

Now the conditions of unidominating function is verified at every vertex.

Sub case 1: Let $i \equiv 0(mod\ 3)$ and $i \neq n$. Then $f(v_i) = 0$.

© 2015, IJARCSSE All Rights Reserved

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 1 + 0 + 0 = 1.$

For i = n, $\sum_{u\in N[v_n]} f(u) = f(v_{n-1}) + f(v_n) + f(v_1) = 1 + 0 + 0 = 1,$

Sub case 2: Let $i \equiv 1(mod\ 3)$ and $i \neq 1$. Then $f(v_i) = 0$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 0 + 0 + 1 = 1.$

For $i = 1$ we have $\sum_{u\in N[v_1]} f(u) = f(v_n) + f(v_1) + f(v_2) = 0 + 0 + 1 = 1.$

Sub case 3: Let $i \equiv 2(mod\ 3)$. Then $f(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 0 + 1 + 0 = 1.$

Since $\sum_{u\in N[v_i]} f(u) = 1$ for $f(v_i) = 1$ *and* $\sum_{u\in N[v_i]} f(u) = 1$ *for* $f(v_i) = 0,$

it follows that f is a unidominating function.

Now $$\sum_{u\in V} f(u) = \underbrace{0+1+0}+\cdots+\underbrace{0+1+0} = \frac{n}{3}.$$

Hence by the definition of unidomination number, $\gamma_u(C_n) \leq \frac{n}{3} - - - (1)$

We know that degree of each vertex of a cycle is equal to two.

If f is a unidominating function of C_n, then we can see that amongst three consecutive vertices in C_n at most two vertices can have functional value 0 and at least one vertex must have functional value 1. Therefore the functional value sum of three consecutive vertices is greater than or equal to 1.

That is $\sum_{i=1}^{3} f(v_i) \geq 1,\ \sum_{i=4}^{6} f(v_i) \geq 1, \ldots, \sum_{i=n-2}^{n} f(v_i) \geq 1.$

Therefore $$f(V) = \sum_{i=1}^{3} f(v_i) + \sum_{i=4}^{6} f(v_i) + \cdots + \sum_{i=n-2}^{n} f(v_i) \geq \underbrace{1+1+\cdots+1}_{\left(\frac{n}{3}\right)-times} \geq \frac{n}{3}.$$

This is true for any unidominating function.

Therefore $\min\{f(V)/f \text{ is a unidominating function}\} \geq \frac{n}{3}.$

Thus $\gamma_u(C_n) \geq \frac{n}{3} - - - (2)$

Therefore from the inequalities (1) and (2), it follows that $\gamma_u(C_n) = \frac{n}{3}$.

Case 2: Let $n \equiv 1(mod\ 3)$.

Define a function $f : V \to \{0,1\}$ *by*

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2(mod\ 3), \\ 0 & for\ i \equiv 0,1(mod\ 3), i \neq n-1, \end{cases}$$

and $f(v_{n-1}) = 1$.

We can see as in Case 1 that f is a unidominating function and

$$\sum_{u\in V} f(u) = \underbrace{0+1+0}+\cdots+\underbrace{0+1+0}+\underbrace{0+1+1+0} = \frac{n-4}{3} + 2 = \frac{n+2}{3} = \left\lceil\frac{n}{3}\right\rceil.$$

By the definition of unidomination number, we have $\gamma_u(C_n) \leq \left\lceil\frac{n}{3}\right\rceil - - - (1)$

Now $n \equiv 1(mod\ 3) \Rightarrow n - 4 \equiv 0(mod\ 3)$. Let f be a unidominating function of

C_n, $n \geq 7$. Then as in Case 1, for any $n-4$ consecutive vertices, we have

$$\sum_{i=1}^{n-4} f(v_i) \geq \frac{n-4}{3}.$$

To get minimum weight, we take the equality in the above inequation. The possibilities of assigning the functional values for the remaining four vertices are as follows:

1,0,0,1 or 0,1,1,0 or 0,0,1,1.

We can see that for any other assignment of functional values to these vertices can no more make f a unidominating function with minimum weight.

$$\text{Now } f(V) = \sum_{u \in V} f(u) = \frac{n-4}{3} + 2 = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil.$$

Thus min $\{f(V)/f \text{ is a unidominating function}\} \geq \left\lceil \frac{n}{3} \right\rceil$.

$$\text{That is } \gamma_u(C_n) \geq \left\lceil \frac{n}{3} \right\rceil --- (2)$$

Therefore from the inequalities (1) and (2), we have $\gamma_u(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

For $n = 4$, the functional values to the 4 vertices can be assigned consecutively as 0,1,1,0 so that

$$\sum_{i=1}^{4} f(v_i) = 2 = \left\lceil \frac{4}{3} \right\rceil.$$

Case 3: Let $n \equiv 2(mod\ 3)$.

Define a function $f: V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2(mod\ 3), i \neq n\ , \\ 0 & \text{for } i \equiv 0,1(mod\ 3), \mathrm{i} \neq \mathrm{n}-2, \mathrm{n}-1\ , \end{cases}$$

and $f(v_{n-2}) = 1, f(v_{n-1}) = 1, f(v_n) = 0.$

We can easily verify that f is a unidominating function.

$$\text{Now} \sum_{u \in V} f(u) = \underbrace{0+1+0}+\cdots+\underbrace{0+1+0}+\underbrace{0+1+1+1+0} = \frac{n-5}{3} + 3 = \frac{n+4}{3}$$

$$= \left\lceil \frac{n}{3} \right\rceil + 1.$$

Therefore by the definition of unidomination number, $\gamma_u(C_n) \leq \left\lceil \frac{n}{3} \right\rceil + 1 --- (1)$

Now $n \equiv 2(mod\ 3) \Longrightarrow n-5 \equiv 0(mod\ 3)$. Then as in Case 2, for any unidominating function f of C_n, $n \geq 8$, with minimum weight we have

$$\sum_{i=1}^{n-5} f(v_i) = \frac{n-5}{3}.$$

The possibilities of assigning the functional values for the remaining five vertices are as follows: 1,1,1,0,0 or 0,1,1,1,0 or 0,0,1,1,1. It can be seen that for any other assignment of functional values to these vertices can no more make f a unidominating function with minimum weight.

$$\text{Now } f(V) = \sum_{u \in V} f(u) = \frac{n-5}{3} + 3 = \frac{n+4}{3} = \left\lceil \frac{n}{3} \right\rceil + 1.$$

Thus min $\{f(V)/f \text{ is a unidominating function}\} \geq \left\lceil \frac{n}{3} \right\rceil + 1$.

$$\text{That is } \gamma_u(C_n) \geq \left\lceil \frac{n}{3} \right\rceil + 1 --- (2)$$

Therefore from the inequalities (1) and (2), $\gamma_u(C_n) = \left\lceil \frac{n}{3} \right\rceil + 1$.

For $n = 5$, we assign the functional values to 5 vertices consecutively as 0,1,1,1,0, so that

$$\sum_{i=1}^{5} f(v_i) = 3 = \left\lceil \frac{5}{3} \right\rceil + 1.$$

Theorem 3.2: The number of unidominating functions of C_n with minimum weight is

$$\begin{cases} 3 & when\ n \equiv 0(mod\ 3), \\ n & when\ n \equiv 1(mod\ 3), \\ n\left(1+\left\lfloor\frac{n}{6}\right\rfloor\right) & when\ n \equiv 2(mod\ 3), n \neq 8, \\ 12 & when\ n = 8. \end{cases}$$

Proof: Let C_n be a cycle with vertex set $V = \{v_1, v_2, \ldots, v_n\}, n \geq 3$.

Now we find the number of unidominating functions of C_n with minimum weight in the following three cases.

Case 1: Let $n \equiv 0(mod\ 3)$.

Let f be a unidominating function defined in Case 1 of Theorem 3.1. Then the functional values of f are $010010 - - - 010$ circularly.

Take $a - 010$. Then the functional values of f are in the pattern of $aa \ldots . a$ circularly (here there are $\frac{n}{3} a's$).

These letters can be arranged circularly in one and only one way. Therefore there is only one unidominating function.

By rotating the functional values of f circularly, we get the possible unidominating functions. Here for two circular rotations we get two other unidominating functions of same weight respectively and the third circular rotation coincides with the given unidominating function f.

Therefore there are three unidominating functions for C_n with minimum weight $\left\lceil\frac{n}{3}\right\rceil$.

Case 2: Let $n \equiv 1(mod\ 3)$.

Let f be a unidominating function defined in Case 2 of Theorem 3.1. Then the functional values of f are $010010 - - - 0100110$ circularly.

Take $a - 010$, $b - 0110$. Then the functional values of f are in the pattern of $aaa \ldots ab$ circularly (here there are $\frac{n-4}{3} a's$). As these letters can be arranged circularly in one and only one way, there exists only one unidominating function.

By rotating the functional values of f circularly, for $n-1$ such circular rotations we get $n-1$ other unidominating functions of same weight. The n^{th} circular rotation coincides with the given unidominating function.

Therefore there are n unidominating functions with minimum weight $\left\lceil\frac{n}{3}\right\rceil$ for C_n.

Case 3: Let $n \equiv 2(mod\ 3)$.

Let f be a unidominating function defined in Case 3 of Theorem 3.1. Then the functional values of f are $010010 - - - 01001110$ circularly.

By taking $a - 010$, $c - 01110$, the functional values of f are in the pattern of $aaa \ldots ac$ circularly (here there are $\frac{n-5}{3}$ $a's$). As in Case 2 these letters can be arranged circularly in only one way. Therefore there exists only one unidominating function f with minimum weight.

By rotating the functional values of f circularly as in Case 2 we get n unidominating functions with minimum weight.

We further investigate some more unidominating functions of C_n of same weight in the following way.

Define a function $f_1 : V \to \{0,1\}$ by

$f_1(v_i) = f(v_i) \quad \forall v_i \in V$ for $i \neq n-5, n-3$,

and $f_1(v_{n-5}) = 1$, $f_1(v_{n-3}) = 0$, $n \geq 8$.

We can easily verify that f_1 is a unidominating function.

Further,

$$\sum_{u \in V} f_1(u) = \underbrace{0+1+0}+\cdots+\underbrace{0+1+0}+\underbrace{0+1+1+0}+\underbrace{0+1+1+0} = \frac{n-8}{3}+4 = \frac{n+4}{3}$$

$$= \left\lceil\frac{n}{3}\right\rceil + 1.$$

The functional values of f_1 are $010010 - - - - 01001100110$ circularly.

Take $a - 010$, $b - 0110$. Then the functional values of f_1 are in the pattern of $aa \ldots abb$ circularly. Here there are $\frac{n-8}{3}$ a's and two $b's$.

That is $\frac{n-8}{3} + 2 = \frac{n-2}{3}$ letters.

If $\frac{n-2}{3}$ is even then these letters can be arranged in $\frac{1}{2}.\frac{n-2}{3} = \frac{n-2}{6} = \left\lfloor\frac{n}{6}\right\rfloor$ ways and if $\frac{n-2}{3}$ is odd then the number of arrangements are $\frac{1}{2}\left[\frac{n-2}{3} - 1\right] = \frac{n-5}{6} = \left\lfloor\frac{n}{6}\right\rfloor$.

Thus there exist $\left\lfloor \frac{n}{6} \right\rfloor$ such unidominating functions.

By rotating the functional values of each such unidominating function we get n unidominating functions when $n \neq 8$. That is we get a total of $n.\left\lfloor \frac{n}{6} \right\rfloor$ unidominating functions when $n \neq 8$.

Hence there are $\left(n + \text{n}.\left\lfloor \frac{n}{6} \right\rfloor\right)$ unidominating functions with minimum weight.

If $n = 8$, then the functional values are 0,1,1,0,0,1,1,0 and after three circular rotations, we get the same function f_1. Hence for $n = 8$, we get 4 unidominating functions.

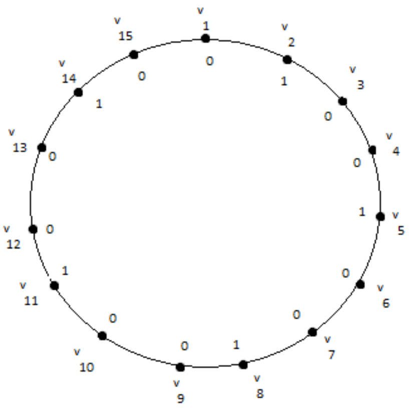
Therefore there are $8 + 4 = 12$ unidominating functions with minimum weight when $n = 8$.

IV. ILLUSTRATIONS

Example 4.1: Let $n = 15$.

Obviously $15 \equiv 0(mod\ 3)$.

The functional values of a unidominating function f defined in Case 1 of Theorem 3.1 are given at the corresponding vertices of C_{15}.



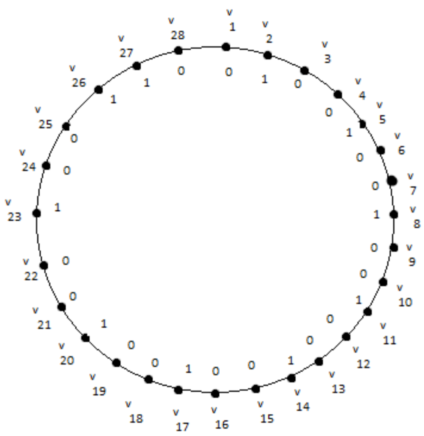
Unidomination number of C_{15} is $\gamma_u(C_{15}) = \left\lceil \frac{15}{3} \right\rceil = 5$.

There are 3 unidominating functions with weight 5.

Example 4.2: Let $n = 28$.

Clearly $28 \equiv 1(mod\ 3)$.

The functional values of a unidominating function f defined in Case 2 of Theorem 3.1 are given at the corresponding vertices of C_{28}.



Unidomination number of C_{28} is $\gamma_u(C_{28}) = \left\lceil \frac{28}{3} \right\rceil = 10$.

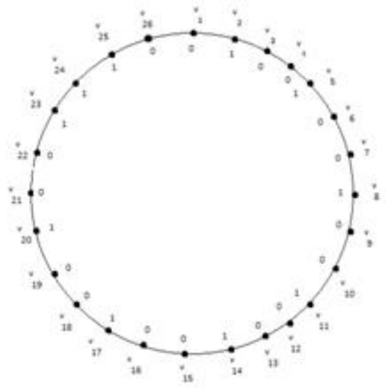
There exists 28 unidominating functionsof weight 10.

Example 4.3: Let $n = 26$.

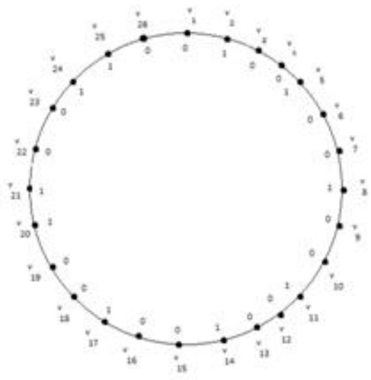
Obviously $26 \equiv 2(mod\ 3)$.

The functional values of two unidominating functions f and f_1 in which f is defined as in Case 3 of Theorem 3.1 and f_1 is defined as in Case 3 of Theorem 3.2 are given at the corresponding vertices.

© 2015, IJARCSSE All Rights Reserved



f



f_1

Unidomination number of C_{26} is $\gamma_u(C_{26}) = \left\lceil \frac{26}{3} \right\rceil + 1 = 10.$

There are $n\left(1+\left\lfloor \frac{n}{6} \right\rfloor\right) = 26\left(1+\left\lfloor \frac{26}{6} \right\rfloor\right) = 26(1+4) = 130$ unidominating functions with weight 10.

V. CONCLUSION

It is interesting to discuss the newly introduced concepts of the unidomination number of a cycle and determining the number of unidominating functions of minimum weight. It may play a vital role in the theory of domination. The authors have also studied the total unidominating functions of cycles.

REFERENCES

[1] T.W.Haynes, S.T.Hedetniemi, and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[2] T.W.Haynes, S.T.Hedetniemi, and P.J.Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[3] Hedetniemi S.M, Hedetniemi, S.T. and Wimer, T.V. – *Linear time resource allocation algorithms for trees*. Technical report URI – 014, Department of Mathematics, Clemson University, 1987.

[4] Anantha Lakshmi, V and Maheswari, B. – *Unidominating Functions of a Path*, IJCET, pp. 11-19, vol 6, 2015.

© 2015, IJARCSSE All Rights Reserved