Complementary Tree Domination Number of Interval Graphs

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Abstract: Interval graphs have found applications in a wide range of fields such as scheduling and genetics, among others. In this paper, we put forth the findings related to complementary tree domination number of interval graphs. The exact value of complementary tree domination number and minimal complementary tree domination sets of some particular classes of interval graphs are obtained.

Mathematics Subject Classification: 05C69

Key words: Interval graphs, domination number, complementary tree domination number

1. Introduction

Graphs considered in this paper are all undirected, connected and simple graphs. Throughout this paper, for the graph \( G = (V, E) \) and for \( S \subseteq V \), the subgraph of \( G \) induced by the vertices in \( S \) is denoted by \( \langle S \rangle \). For any vertex \( v \in V \) \((G)\), \( N(v) \) denotes the open neighborhood of \( v \) and is defined as the set of all vertices adjacent to \( v \) in \( G \) and \( N \{v\} \) denotes the closed neighborhood of \( v \) and is defined as \( N \{v\} = N(v) \cup \{v\} \). A vertex of degree one is called a support.

A subset \( S \) of the vertex set \( V \) of the graph \( G = (V, E) \) is a dominating set of the graph \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of the graph \( G \) denoted by \( \gamma(G) \) and is the minimum cardinality of a dominating set of \( G \). A dominating set \( S \subseteq V \) of a graph \( G \) with vertex set \( V \) \((G) \) and edge set \( E \) \((G) \) is a complementary tree domination set if the induced subgraph \( \langle V - S \rangle \) is a tree. Complementary tree domination number is the minimum cardinality of a complementary tree dominating set of \( G \). It is denoted by \( \gamma_{ctd}(G) \). The notion of dominating set is due to Ore [1], Cockayne and Hedetnieme [2] contributed to the domination theory in graphs besides many others. For more details about the domination number one can refer to Walikar et al. [3]. The notion of complementary tree dominating set is due to S. Muttamai et al. [4]. Some results pertaining to the bounds of Complementary tree domination number are obtained by them.

Interval graphs are a special class of circular-arc graphs that can be represented with a set of arcs that do not cover the entire circle. The extensive study of interval graphs has been done for several decades by both mathematicians and computer scientists.

Let \( I = \{I_1, I_2, I_3, \ldots, I_k\} \) be an interval family, where each \( I_i \) is an interval on the real line and \( I_i = [a_i, b_i] \), for \( i = 1, 2, 3, \ldots, k \). Here \( a_i \) is called the left end point and \( b_i \) is called the right end point. Without loss of generality, one can assume that, all end points of the intervals are distinct numbers between 1 and 2\( k \). The intervals are named in the increasing order of their right end points. The graph \( G = (V, E) \) is an interval graph if there is one-to-one correspondence between the vertex set \( V \) and the interval family \( I \). Two vertices of \( G \) are joined by an edge if and only if their corresponding intervals in \( I \) intersect. That is if \( I_1 = [a_1, b_1] \) and \( I_2 = [a_2, b_2] \), then \( I_1 \) and \( I_2 \) will intersect if \( a_1 < b_2 \) or \( a_2 < b_1 \). Interval graphs are rich in combinatorial structures and have found applications in several disciplines such as traffic control, ecology, biology, computer sciences and particularly useful in cyclic scheduling and computers storage allocation problems etc. Having a representation of graph with intervals or arcs can be helpful in combinatorial problems of the graph, such as isomorphism testing and finding maximum independent set and cliques of graphs.

2. Observations

Observation 2.1: Bounds of complementary tree domination number
Let \( G \) be a connected interval graph of order \( k \geq 2 \). Then \( \gamma_{ctd}(G) \leq k-1 \)

Observation 2.2: Relation between domination number and complementary tree domination number
Let \( G \) be a connected interval graph of order \( k \geq 2 \). Then \( \gamma(G) \leq \gamma_{ctd}(G) \)

For any graph \( G \), every complementary tree dominating set is a dominating set. But every dominating set need not be a complementary tree dominating set. Hence the result follows.

3. Important Results

The following results obtained by S. Muttamai et al. [4] characterize ctd-sets.

Result 3.1: Every pendant vertex is a member of all ctd-sets.

Result 3.2: A ctd-set \( S \) of \( G \) is minimal if and only if for each vertex \( v \) in \( S \), one of the following conditions holds.

(i) \( v \) is an isolated vertex of \( S \).

(ii) There exists a vertex in \( V - S \) for which \( N(u) \cap S = \{v\} \).
(iii) \( N(v) \cap (V - D) = \emptyset \)

(iv) The induced subgraph \(< V - S \cup \{v\}>\) either contains a cycle or disconnected.

4. Complementary Tree Domination Number Of Interval Graphs

In this section, the exact values of complementary tree domination number and minimal complementary tree domination sets of some particular classes of interval graphs are obtained.

**Theorem 4.1:** Let \( I = \{I_1, I_2, \ldots, I_k\}, k \geq 2 \) be an interval family corresponding to an interval graph \( G \). Suppose that there exists an interval \( I_i \in I \) such that every other interval of the family can’t but simply dominate any other interval except the interval \( I_i \), then

\[
\gamma_{ctd}(G) = k-1
\]

Proof: Let \( v_i \) be the vertex corresponding to the interval \( I_i \) respectively for \( i = 1, 2, 3, \ldots, k \). Let the interval family \( I = \{I_1, I_2, \ldots, I_n\}, k \geq 2 \) satisfy the condition mentioned in the theorem. Then the vertices \( v_1, v_2, v_3, \ldots, v_k \) are pendant vertices as they are adjacent to one and only one vertex \( v_i \). But every pendant vertex is a member of all ctd-sets. Thus

For any connected graph \( G \) of order \( k \geq 2 \),

\[
\gamma_{ctd}(G) \geq k-1 \quad \text{(1)}
\]

\[
\gamma_{ctd}(G) \leq k-1 \quad \text{(2)}
\]

From (1) and (2). It follows that

\[
\gamma_{ctd}(G) = k-1
\]

and minimal complementary tree dominating set is \( \{v_1, v_2, v_3, \ldots, v_k\} \).

**Illustration 4.1.1:** Let the interval family \( I = \{1, 2, 3, \ldots, 7\} \) corresponding to the interval graph \( G \) be as follows:

- Interval family:
  - 1
  - 2
  - 3
  - 6
  - 4
  - 5
  - 7

Clearly the interval family satisfies the conditions mentioned in the theorem 4.1 for \( k=7 \). Therefore the complementary tree domination number of the graph \( G = k-1 = 7-1 = 6 \). Minimal complementary tree dominating set is \( \{1, 2, 3, 4, 5, 7\} \).

**Theorem 4.2:** Let \( I = \{I_1, I_2, \ldots, I_k\}, k \geq 4 \) be an interval family analogous to an interval graph \( G \). In a condition, wherein the intervals \( I_1, I_2 \) are intersecting intervals and any interval of \( I \) other than \( I_1 \) and \( I_2 \) doesn’t dominate any other interval except \( I_1 \), or \( I_2 \), but not both, then

1. \( \gamma_{ctd}(G) = k-2 \) with minimal ctd-set \( I - \{I_1, I_2\} \), if some of the intervals in \( I-\{I_1, I_2\} \) dominate \( I_1 \) and some of the intervals in \( I-\{I_1, I_2\} \) dominate \( I_2 \\

2. \( \gamma_{ctd}(G) = k-1 \), if all the intervals in \( I-\{I_1, I_2\} \) dominate \( I_1 \) or all the intervals in \( I-\{I_1, I_2\} \) dominate \( I_2 \\

**Proof:** Let the interval family \( I = \{I_1, I_2, \ldots, I_k\}, k \geq 4 \) satisfy the conditions mentioned in the hypothesis. Here two cases may arise

Case (i): Some of the intervals in \( I-\{I_1, I_2\} \) may dominate \( I_1 \) and the remaining intervals in \( I-\{I_1, I_2\} \) may dominate \( I_2 \). Let \( v_1, v_2, v_3, \ldots, v_k \) be the vertices corresponding to the intervals \( I_1, I_2, \ldots, I_k \) respectively. By the conditions for domination between the intervals of the interval family \( I \), it is clear that except the vertices \( v_i \) and \( v_j \), the remaining \( k-2 \) vertices are pendant vertices. Every pendant vertex is a member of complementary tree dominating set. Therefore

\[
\gamma_{ctd}(G) \geq k-2 \quad \text{(1)}
\]

\[
\gamma_{ctd}(G) \leq k-2 \quad \text{(2)}
\]

From (1) and (2), it is clear that \( \gamma_{ctd}(G) = k-2 \) with minimal ctd-set \( I - \{I_1, I_2\} \).

Case (ii): All the intervals in \( I-\{I_1, I_2\} \) may dominate only the interval \( I_1 \). Then all the intervals of \( I \) other than \( I_1 \) will not dominate any other interval except \( I_1 \). Therefore, all the vertices of the vertex set \( V \) except \( v_i \) are pendant vertices. As a result every ctd-set contains all the vertices except \( v_i \). Implies

\[
\gamma_{ctd}(G) \geq k-1 \quad \text{(1)}
\]

But for every connected graph

\[
\gamma_{ctd}(G) \leq k-1 \quad \text{(2)}
\]

Hence, \( \gamma_{ctd}(G) = k-1 \) with minimal ctd-set \( I - \{I_1\} \).

Case (iii): All the intervals in \( I-\{I_1, I_2\} \) may dominate only the interval \( I_2 \). Then all the intervals of \( I \) other than \( I_2 \) will not dominate any other interval except \( I_2 \). By the similar type of argument as in the previous case, it can be proved that \( \gamma_{ctd}(G) = k-1 \) and the minimal ctd-set in this case as \( I - \{I_2\} \). Hence the theorem.
Illustration 4. 2. 1: Let the interval family $I = \{I_1, I_2, I_3, \ldots, I_7\}$ corresponding to the interval graph $G$ be as follows:

![Graph Image]

**Interval family**

Here, the interval $I_6$ intersects the interval $I_1$ and any interval of $I$ other than $I_6$ and $I_7$ does not dominate any other interval except $I_6$. It follows that conditions mentioned in the theorem 4. 2 are satisfied for $i = 4$ and $j = 5$ (case i). Hence, $\gamma_{\text{ctd}}(G) = k-2 = 7-2 = 5$

**Verification:** Let $v_1, v_2, \ldots, v_7$ be the vertices of the interval graph $G$ corresponding to the intervals $I_1, I_2, \ldots, I_7$ of the interval family $I$ respectively.

Clearly the interval family satisfies the conditions mentioned in the theorem 4.2 (case i) for $k=7$. The graph $G$ corresponding to the interval family $I$ will be as follows:

![Interval Graph Image]

Clearly, $\gamma_{\text{ctd}}(G) = 5$ with minimal ctd-set = $\{v_1, v_2, v_3, v_6, v_7\}$

Illustration 4. 2. 2: Let the interval family $I = \{I_1, I_2, I_3, \ldots, I_7\}$ corresponding to the interval graph $G$ be as follows:

![Graph Image]

**Interval family**

Here, the interval $I_6$ intersects the interval $I_1$, and any interval of $I$ other than $I_6$ and $I_7$ does not dominate any other interval except $I_6$. It follows that conditions mentioned in the theorem 4. 2 are satisfied for $i = 4$ and $j = 7$ (case ii). Hence, $\gamma_{\text{ctd}}(G) = k-1 = 7-1 = 6$ with the set of arcs $\{I_1, I_2, I_3, I_4, I_5, I_7\}$ as the minimal ctd-set.

**Verification:** Let $v_1, v_2, \ldots, v_7$ be the vertices of the interval graph $G$ corresponding to the intervals $I_1, I_2, \ldots, I_7$ of the interval family $I$ respectively.

Clearly the interval family satisfies the conditions mentioned in the theorem 4.2 (case ii) for $k=7$. The interval graph $G$ corresponding to the interval family $I$ will be as follows:

![Interval Graph Image]

Clearly, $\gamma_{\text{ctd}}(G) = 6$ with minimal ctd-set $\{v_1, v_2, v_3, v_4, v_5, v_7\}$.

Theorem 4.3: In the family of intervals $I = \{I_1, I_2, \ldots, I_{2k}\}$ corresponding to the interval graph $G$, suppose that no other intersections are observed except the following:

(i) $I_i$ intersects no other interval except $I_{i+1}$ which contains only the interval $I_i$ for $i = 1, 3, 5, \ldots, 2k-1$ and

(ii) $I_i$ intersects no other non contained intervals except $I_{i-2}$ and $I_{i+2}$ for $i = 4, 6, \ldots, 2(k-1)$

Then, $\gamma_{\text{ctd}}(G) = k$ for $k \geq 3$, where $k \in N$ with the set of intervals $\{I_1, I_3, \ldots, I_{2k-1}\}$ as the minimal ctd-set.

**Proof:** Let the interval family $I = \{I_1, I_3, \ldots, I_{2k}\}$ corresponding to the interval family $G$ satisfy the conditions mentioned in the theorem. By first condition stated in the theorem $I_1$ intersects no other interval except $I_2$. 

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I₁ intersects no other interval except I₅.
I₂ intersects no other interval except I₆.

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I₃k₁ intersects no other interval except I₃k₂.
By the first and second conditions stated in the theorem
I₁ intersects no other interval except I₄ and I₅.
I₁ intersects no other interval except I₃ and I₅.
I₆ intersects no other interval except I₄, I₅ and I₆.

..........................................................................

I₂k₂ intersects no other interval except I₂k₂, I₃k₂, and I₄k₂.
I₃k₂ intersects no other interval except I₂k₂ and I₃k₂.
Let v₁, v₂, v₃, ....... v₂k be the vertices corresponding to the intervals I₁, I₂, ....... I₂k respectively. Every dominating set of G contains either vᵢ or vᵢ₊₁, for i = 1, 3, 5, .... 2k-1. As a result the set of intervals { v₁, v₃, ....... v₂k } and { v₂, v₄, ....... v₂k+₁ } are minimal dominating sets of the graph G. It follows that, γ(G) = K. By the second condition of the theorem, it is clear that the vertices v₁, v₃, ....... v₂k are the only pendant vertices of the interval graph. Every ctd-set contains each pendant vertex of the graph. It follows that

γ_{ctd}(G) ≥ k

Let S = { v₁, v₃, ....... v₂k } Thus V - S = { v₂, v₄, ....... v₂k+₁ }. The set S is a dominating set and the induced subgraph < V - S > is a path graph which is a tree. The set S is a ctd-set. Implies

γ_{ctd}(G) ≤ k

As a result, γ_{ctd}(G) = k with the set of vertices{ v₁, v₃, ....... v₂k-1 } i.e., the set of arcs { I₁, I₃, ....... I₂k-1 } as the minimal ctd-set.

Illustration 4.3.1: Let the interval family I = { I₁, I₂, I₃, ...... I₈ } corresponding to the interval graph G be as follows:

```
   I₁  I₂  I₃  I₄  I₅  I₆  I₇  I₈
```

**Interval family I**

Clearly the interval family satisfies the conditions mentioned in the theorem 4.3 for k=4. Therefore, γ_{ctd}(G) = k = 4. The minimal ctd-set is { I₁, I₅, I₆, I₇ }

**Theorem 4.4:** Let I = { I₁, I₂, I₃, ...... Iₖ } be the interval family corresponding to an interval graph G. For any three consecutive intervals Iᵢ, Iᵢ₊₁ and Iᵢ₊₂ if Iᵢ doesn’t dominate any other interval except Iᵢ₊₁ and Iᵢ₊₂ then

γ_{ctd}(G) = k

**Proof:** Let G be the interval graph, whose interval family I = { I₁, I₂, I₃, ...... Iₖ } satisfies the condition mentioned in the theorem. By the hypothesis,
the interval I₁ intersects the interval I₂.
the interval I₃ intersects the intervals I₁ and I₃.
the interval I₅ intersects the intervals I₃ and I₅.
the interval I₇ intersects the intervals I₅ and I₇.

..........................................................................

the interval Iₙ₋₁ intersects the intervals Iₙ₋₂ and Iₙ.
the interval Iₙ intersects the intervals Iₙ₋₂ and Iₙ.
The vertex vᵢ₁, vᵢ₂, vᵢ₃, ...... vᵢₖ be the vertices corresponding to the intervals I₁, I₂, I₃, ...... Iₖ respectively. Let Sᵢ = { vᵢ, vᵢ₊₁, ....... vᵢ₂k, vᵢ₂k₊₁, ....... vᵢₖ } for i = 2, 3, ...... k-2. Then

V - Sᵢ = { vᵢ, vᵢ₊₁ } for i = 2, 3, ...... k-2.

Since I₁, I₃ and I₅ are three consecutive arcs, I₁ dominates I₃ and as I₁, I₃ and I₅ are three consecutive arcs, I₃ dominates I₅ for i = 2, 3, ...... k-2. Every vertex in V - Sᵢ is adjacent to some vertex in Sᵢ. So the set Sᵢ is a dominating set and also the induced subgraph < V - Sᵢ > consists of only two vertices vᵢ and vᵢ₊₁ with an edge between them. Implies, the induced subgraph < V - Sᵢ > is a tree. Therefore, the set Sᵢ is a ctd-set for each i = 2, 3, ...... k-2. Cardinality of Sᵢ is k-2. It follows that

γ_{ctd}(G) ≤ k - 2.

Any set with cardinality k - 2 other than Sᵢ for i = 2, 3, ...... k-2 is not a ctd-set. Now we shall show that any subset of Sᵢ is not a dominating set. Here three cases will arise.

**Case (i):** Let Sᵢ = Sᵢ₋₁ \ { vᵢ₋₁ }. In this case again two sub cases will arise.

**Sub case (i):** Let i ≠ 2. Then vᵢ is neither adjacent to vᵢ₋₁ nor vᵢ₊₁. The induced subgraph < V - Sᵢ > consists of isolated vertex vᵢ. The induced subgraph < V - Sᵢ > is a disconnected graph and hence not a tree. Therefore, the set Sᵢ is not a ctd-set.
Sub case (ii): Let \( i = 2 \). Then \( S_i' = S_i - \{v_1\} = \{v_4, v_5, \ldots, v_k\} \), where \( i = 2 \). In this case, the induced subgraph \( \langle V - S_i' \rangle \) is a tree with vertex set \( \{v_1, v_2, v_3\} \) and edges \( \{v_1, v_2\}, \{v_2, v_3\} \) and \( \{v_1, v_3\} \). But the set \( S_i' \) is not a dominating set since the vertex \( v_3 \) in the set \( V - S_i' \) is not adjacent to none of the vertices in \( S_i' \). Thus the set \( S_i' \) is not a ctd-set. In both the cases, \( S_i' \) is not a ctd-set.

Case (ii): Let \( S_i = S_i - \{v_k\} \). When \( i \neq k - 2 \), the induced subgraph \( \langle V - S_i' \rangle \) is not a tree and when \( i = k - 2 \), the set \( S_i' \) is not a dominating set. Hence in both the cases \( S_i' \) is not a ctd-set.

Case (iii): Let \( S_i = S_i - \{v_j\} \), where \( j \neq 1, k \). With the similar type of argument it can be proved that \( S_i' \) is not a ctd-set. From all the possible three cases, it is clear that any sub set of \( S_i \) is not a ctd-set. Implies

\[
\gamma_{\text{ctd}}(G) \geq k - 2
\]

Hence, \( \gamma_{\text{ctd}}(G) = k - 2 \). In this case, the interval family has more than one minimal ctd-set. Minimal ctd-sets are given by \( S_i = \{v_1, v_2, \ldots, v_{i-1}, v_{i+2}, v_{i+3}, \ldots, v_k\} \) i.e., the family \( \{I_1, I_2, \ldots, I_{i-1}, I_{i+2}, I_{i+3}, \ldots, I_k\} \), where \( i = 2, 3, \ldots, k - 2 \).

Illustration 4.4.1: Let the interval family \( I = \{I_1, I_2, I_3, \ldots, I_8\} \) corresponding to the interval graph \( G \) be as follows:

![Fig. a: Interval family](image)

Clearly the interval family \( I \) satisfy the conditions mentioned in the theorem 4.4 for \( k = 8 \). Therefore, \( \gamma_{\text{ctd}}(G) = k - 2 = 8 - 2 = 6 \) with one of the minimal ctd-set as \( \{1, 2, 3, 4, 7, 8\} \)

References